Solving the time dependent Ginzburg-Landau equation using COMSOL Multiphysics

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Outline

1. Ginzburg-Landau theory
2. Review of numerical simulations of TDGL equation
3. TDGL equation in COMSOL
4. PDE simulation flowchart

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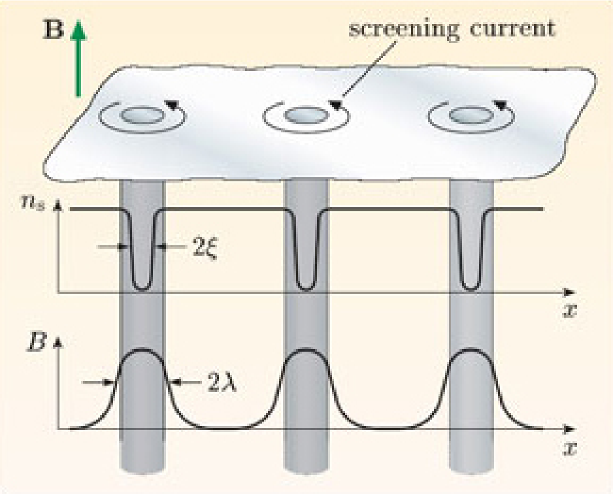
Type-II superconductors



Superconductivity at open.edu

The time dependent Ginzburg-Landau (TDGL) equation

$$\frac{\partial \Psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \Psi + \Psi - |\Psi|^2 \Psi \tag{1}$$

$$\sigma \frac{\partial \boldsymbol{A}}{\partial t} = \frac{1}{2i\kappa}\left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*\right) - |\Psi|^2 \boldsymbol{A} - \nabla \times \left(\nabla \times \boldsymbol{A} - \boldsymbol{B}_a\right) \tag{2}$$

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Boundary conditions

$$\nabla \Psi \cdot \boldsymbol{n} = 0, \quad \text{on } \partial\Omega \qquad (3)$$

$$\nabla \times \boldsymbol{A} = \boldsymbol{B}_a, \quad \text{on } \partial\Omega \qquad (4)$$

$$\boldsymbol{A} \cdot \boldsymbol{n} = 0, \quad \text{on } \partial\Omega \qquad (5)$$

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Numerical solution of TDGLE

Finite difference approximation

- W.D. Gropp et al. *J. Comp. Phys.*, 123 (1996) 254.
- M.V. Milošević, *Physica C*, 470 (2010) 791.
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Numerical Simulation of Vortex Dynamics in Type-II Superconductors*

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Received November 14, 1994; revised April 10, 1995

This article describes the results of several numerical simulations of vortex dynamics in type-II superconductors. The underlying mathematical model is the time-dependent Ginzburg–Landau model. The simulations concern vortex penetration in the presence of twin boundaries, interface patterns between regions of opposite vortex orientation, and magnetic-flux entry patterns in superconducting samples.

1. GINZBURG–LANDAU MODEL

In this article we report on several numerical simulations of vortex motion in type-II superconducting materials [1, 2]. The simulations are based on the time-dependent Ginzburg–Landau (TDGL) equations,

$$\mathbf{J}_s = \frac{e_s\hbar}{2im_s}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{e_s^2}{m_s c}|\psi|^2\mathbf{A}. \qquad (1.4)$$

The superscript * denotes complex conjugation, e_s is the "effective charge" of a superelectron ($e_s < 0$), and m_s its "effective mass." The constant D is a phenomenological diffusion coefficient. If $\mathbf{J}$ is viewed as the sum of a "normal" current, which satisfies Ohm's law, and the supercurrent, σ may be interpreted as the "coefficient of normal conductivity."

The quantities a and b in (1.3) are phenomenological parameters; they are functions of external parameters, such as the temperature T, the concentration of impurities, etc.; $b > 0$ for all T, and a changes sign at T_c ($a < 0$ for $T < T_c$, $a > 0$ for $T > T_c$).

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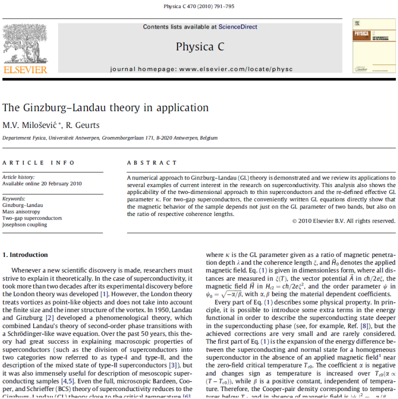

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The Ginzburg–Landau theory in application

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A R T I C L E I N F O

Article history:
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Keywords:
Ginzburg–Landau
Mass anisotropy
Two-gap superconductors
Josephson coupling

A B S T R A C T

A numerical approach to Ginzburg–Landau (GL) theory is demonstrated and we review its applications to several examples of current interest in the research on superconductivity. This analysis also shows the applicability of the two-dimensional approach to thin superconductors and the re-defined effective GL parameter κ. For two-gap superconductors, the conveniently written GL equations directly show that the magnetic behavior of the sample depends not just on the GL parameter of two bands, but also on the ratio of respective coherence lengths.

1. Introduction

Whenever a new scientific discovery is made, researchers must strive to explain it theoretically. In the case of superconductivity, it took more than two decades after its experimental discovery before the London theory was developed [1]. However, the London theory treats vortices as point-like objects and does not take into account the finite size and the inner structure of the vortex. In 1950, Landau and Ginzburg [2] developed a phenomenological theory, which combined Landau's theory of second-order phase transitions with a Schrödinger-like wave equation. Over the past 50 years, this theory had great success in explaining macroscopic properties of superconductors (such as the division of superconductors into two categories now referred to as type-I and type-II, and the description of the mixed state of type-II superconductors [3]), but it was also immensely useful for description of mesoscopic superconducting samples [4,5]. Even the full, microscopic Bardeen, Cooper, and Schrieffer (BCS) theory of superconductivity reduces to the

where κ is the GL parameter given as a ratio of magnetic penetration depth λ and the coherence length ξ, and $\vec{H}_0$ denotes the applied magnetic field. Eq. (1) is given in dimensionless form, where all distances are measured in $\xi(T)$, the vector potential $\vec{A}$ in $ch/2e\xi$, the magnetic field $\vec{H}$ in $H_{c2} = ch/2e\xi^2$, and the order parameter ψ in $\psi_0 = \sqrt{-\alpha/\beta}$, with α, β being the material dependent coefficients.

Every part of Eq. (1) describes some physical property. In principle, it is possible to introduce some extra terms in the energy functional in order to describe the superconducting state deeper in the superconducting phase (see, for example, Ref. [8]), but the achieved corrections are very small and are rarely considered. The first part of Eq. (1) is the expansion of the energy difference between the superconducting and normal state for a homogeneous superconductor in the absence of an applied magnetic field[1] near the zero-field critical temperature T_{c0}. The coefficient α is negative and changes sign as temperature is increased over $T_{c0}(\alpha \propto (T - T_{c0}))$, while β is a positive constant, independent of temperature. Therefore, the Cooper-pair density corresponding to tempera-

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JOURNAL OF MATHEMATICAL PHYSICS **46**, 095109 (2005)

Numerical approximations of the Ginzburg–Landau models for superconductivity

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(Received 16 May 2005; accepted 17 May 2005; published online 29 September 2005)

In this paper, we review various methods for the numerical approximations of the Ginzburg–Landau models of superconductivity. Particular attention is given to the different treatment of gauge invariance in both the finite element, finite difference, and finite volume settings. Representative theoretical results, typical numerical simulations, and computational challenges are presented. Generalizations to other relevant models are also discussed. © *2005 American Institute of Physics.* [DOI: 10.1063/1.2012127]

I. INTRODUCTION

The macroscopic model of Ginzburg and Landau[43,91] has been widely used to study both low-temperature and high-temperature superconductors. Due to its highly nonlinear nature, the complex energy landscape and the exotic dynamic responses of its solution to external conditions, its numerical simulations have become valuable tools in order to better understand the properties of the Ginzburg–Landau (GL) models and to provide further theoretical insight into the intriguing superconductivity phenomena.

The development of approximation methods of the Ginzburg–Landau model goes back to the

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Pergamon

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Finite Element Methods for the Time-Dependent Ginzburg-Landau Model of Superconductivity

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East Lansing, MI 48824, U.S.A.

(Received and accepted April 1993)

Abstract—The initial-boundary value problem for the time-dependent Ginzburg-Landau equations that model the macroscopic behavior of superconductors is considered. The convergence of finite-dimensional, semidiscrete Galerkin approximations is studied as is a fully-discrete scheme. The results of some computational experiments are presented.

Keywords—Superconductivity, Time-dependent Ginzburg-Landau equations, Finite element methods.

1. THE TIME-DEPENDENT GINZBURG-LANDAU EQUATIONS

The steady state Ginzburg-Landau model for superconductivity (see, e.g., [1] or [2]) was extended to the time-dependent case by Gor'kov and Eliashberg in [3]. The latter model is defined by the differential equations

$$\eta \frac{\partial \psi}{\partial t} + i\eta\kappa\Phi\psi + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi - \psi + |\psi|^2\psi = 0, \qquad \text{in } \Omega \times [0,T], \qquad (1.1)$$

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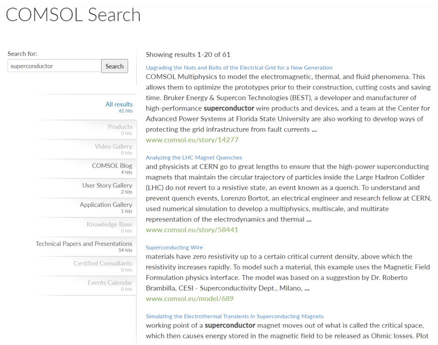

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DOI 10.1007/s10440-010-9580-8

Magnetic Flux Lines in Complex Geometry Type-II Superconductors Studied by the Time Dependent Ginzburg-Landau Equation

Tommy Sonne Alstrøm · Mads Peter Sørensen · Niels Falsig Pedersen · Søren Madsen

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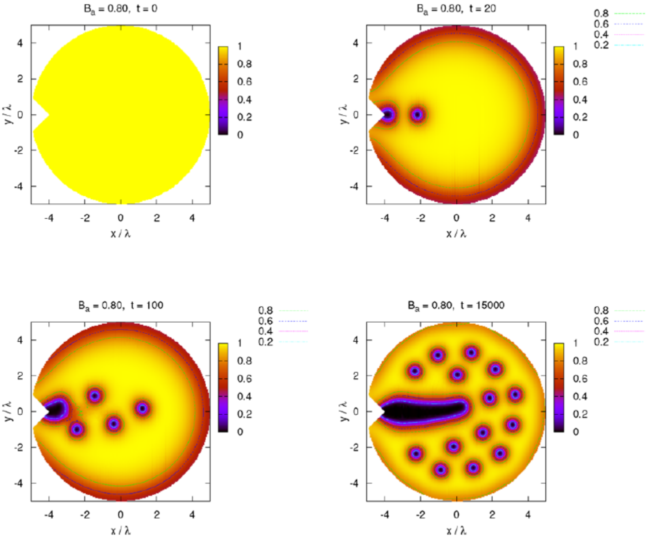
Abstract The time-dependent Ginzburg-Landau equation is solved numerically for type-II superconductors of complex geometry using the finite element method. The geometry has a marked influence on the magnetic vortex distribution and the vortex dynamics. We have observed generation of giant vortices at boundary defects, suppressing the superconducting state far into the superconductor.

Keywords Type II superconductivity · Ginzburg-Landau equation · Vortex lattices · Giant vortices

1 Introduction

In 1950 V.L. Ginzburg and L.D. Landau proposed a phenomenological theory for superconducting phase transitions [1]. The theory is based on a Schrödinger equation with a ϕ-4 potential and a kinetic term involving the momentum operator. For type-II superconductors the Ginzburg-Landau equation models the magnetic field penetration through quantized current vortices as the externally applied magnetic field exceeds a threshold value. A number

Vortex dynamics in type-II superconductors



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General form PDE interface

General form PDE

$$\boldsymbol{e}_a \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \boldsymbol{d}_a \frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{\Gamma} = \boldsymbol{F}$$

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Dirichlet boundary conditions

$$\boldsymbol{R} = \mathbf{0}, \text{ on } \Omega$$

$\boldsymbol{R} = \boldsymbol{R}(x, y, t, \boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y)$

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Neumann boundary conditions

$$-\boldsymbol{n} \cdot \boldsymbol{\Gamma} = \boldsymbol{G} + \left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{u}}\right)^T \cdot \boldsymbol{\mu}, \text{ on } \Omega$$

$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(x, y, t, \boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y)$, $G = G(x, y, t, \boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y)$

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Zero flux

$$-\boldsymbol{n} \cdot \boldsymbol{\Gamma} = 0, \text{ on } \Omega$$

Notation

$$\Psi = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \nabla = \begin{pmatrix} \partial_x & \partial_y \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}$$

The Eq. (1) $\frac{\partial \Psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \Psi + \Psi - |\Psi|^2 \Psi$ we can rewrite as:

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\left(\frac{i}{\kappa}\nabla + A\right)^2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - (u_1^2 + u_2^2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

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$$\partial_t u_1 = \nabla \cdot \begin{pmatrix} u_{1x}/\kappa^2 \\ u_{1y}/\kappa^2 \end{pmatrix} + \frac{1}{\kappa}\left(u_{3x} + u_{4y}\right) u_2 + \frac{2}{\kappa}\left(u_3 u_{2x} + u_4 u_{2y}\right) - (u_3^2 + u_4^2) u_1 + u_1 - (u_1^2 + u_2^2) u_1$$

$$\partial_t u_2 = \nabla \cdot \begin{pmatrix} u_{2x}/\kappa^2 \\ u_{2y}/\kappa^2 \end{pmatrix} - \frac{1}{\kappa}\left(u_{3x} + u_{4y}\right) u_1 - \frac{2}{\kappa}\left(u_3 u_{1x} + u_4 u_{1y}\right) - (u_3^2 + u_4^2) u_2 + u_2 - (u_1^2 + u_2^2) u_2$$

$$\partial_t u_1 \quad + \quad \nabla \quad \cdot \quad \underbrace{\begin{pmatrix} -u_{1x}/\kappa^2 \\ -u_{1y}/\kappa^2 \end{pmatrix}}_{\Gamma_1} \quad = \quad F_1$$

$$F_1 = -\frac{1}{\kappa}\left(u_{3x} + u_{4y}\right)u_1 - \frac{2}{\kappa}\left(u_3 u_{1x} + u_4 u_{1y}\right) - (u_3^2 + u_4^2)u_2$$

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$$F_2 = -(u_3^2 + u_4^2)u_2 - \frac{1}{\kappa}\left((u_{3x} + u_{4y})u_2 + u_3 u_{1x} + u4 u_{1y}\right) + u_2 - (u_1^2 + u_2^2)u_2$$

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$$\partial_t u_2 + \nabla \cdot \underbrace{\begin{pmatrix} -u_{2x}/\kappa^2 \\ -u_{2y}/\kappa^2 \end{pmatrix}}_{\Gamma_2} = F_2$$

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$$\nabla \Psi \cdot \boldsymbol{n} = 0 \quad \equiv \quad -\boldsymbol{n} \cdot \boldsymbol{\Gamma} = 0$$

The Eq. (2) $\sigma\frac{\partial \boldsymbol{A}}{\partial t} = \frac{1}{2i\kappa}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - |\Psi|^2\boldsymbol{A} - \nabla\times(\nabla\times\boldsymbol{A} - \boldsymbol{B}_a)$ we can rewrite as:

$$\sigma\partial_t u_3 = (u_1 u_{2x} - u_2 u_{1x})/\kappa - (u_1^2 + u_2^2)u_3 - \nabla\cdot(0, u_{4x} - u_{3y} - B_{az})^T$$
$$\sigma\partial_t u_4 = (u_1 u_{2y} - u_2 u_{1y})/\kappa - (u_1^2 + u_2^2)u_3 - \nabla\cdot(-u_{4x} + u_{3y} + B_{az}, 0)^T$$

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$$\sigma \partial_t u_3 + \nabla \cdot \underbrace{(0, u_{4x} - u_{3y} - B_{az})^T}_{\Gamma_3} = \underbrace{(u_1 u_{2x} - u_2 u_{1x})/\kappa - (u_1^2 + u_2^2) u_3}_{F_3}$$
$$\sigma \partial_t u_4 + \nabla \cdot \underbrace{(-u_{4x} + u_{3y} + B_{az}, 0)^T}_{\Gamma_4} = \underbrace{(u_1 u_{2y} - u_2 u_{1y})/\kappa - (u_1^2 + u_2^2) u_3}_{F_4}$$

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$$\sigma \partial_t u_3 + \nabla \cdot \underbrace{(0, u_{4x} - u_{3y} - B_{az})^T}_{\Gamma_3} = \underbrace{(u_1 u_{2x} - u_2 u_{1x})/\kappa - (u_1^2 + u_2^2) u_3}_{F_3}$$
$$\sigma \partial_t u_4 + \nabla \cdot \underbrace{(-u_{4x} + u_{3y} + B_{az}, 0)^T}_{\Gamma_4} = \underbrace{(u_1 u_{2y} - u_2 u_{1y})/\kappa - (u_1^2 + u_2^2) u_3}_{F_4}$$

$$\nabla \times \boldsymbol{A} = \boldsymbol{B}_a \quad \equiv \quad -\boldsymbol{n} \cdot \boldsymbol{\Gamma} = 0$$

How to implement the boundary condition:
$\boldsymbol{A} \cdot \boldsymbol{n} = 0, \quad$ on $\partial\Omega$?

How to implement the boundary condition:

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Use dummy function u_5!!!

How to implement the boundary condition:
$\boldsymbol{A} \cdot \boldsymbol{n} = 0, \quad$ on $\partial\Omega$?
Use dummy function u_5!!!

$$\nabla \cdot \underbrace{\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}}_{\Gamma_5} = \underbrace{u_{3x} + u_{4y} + u_5}_{F_5}$$

TDGLE in COMSOL

$$\boldsymbol{e}_a \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \boldsymbol{d}_a \frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{\Gamma} = \boldsymbol{F}$$

$\boldsymbol{e}_a = \boldsymbol{0}$, $\boldsymbol{d}_a = \text{diag}\,\{1, 1, \sigma, \sigma, 0\}$

$$\boldsymbol{\Gamma} = \begin{pmatrix} (-u_{1x}/\kappa^2, -u_{1y}/\kappa^2)^T \\ (-u_{2x}/\kappa^2, -u_{2y}/\kappa^2)^T \\ (0, u_{4x} - u_{3y} - B_{az})^T \\ (-u_{4x} + u_{3y} + B_{az}, 0)^T \\ (u3, u4)^T \end{pmatrix}, \quad \boldsymbol{F} = \begin{pmatrix} F_1 \\ F_2 \\ (u_1 u_{2x} - u_2 u_{1x})/\kappa - (u_1^2 + u_2^2) u_3 \\ (u_1 u_{2y} - u_2 u_{1y})/\kappa - (u_1^2 + u_2^2) u_3 \\ u_{3x} + u_{4y} + u_5 \end{pmatrix}$$

TDGLE in COMSOL

$$\boldsymbol{e}_a \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \boldsymbol{d}_a \frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{\Gamma} = \boldsymbol{F}$$

$\boldsymbol{e}_a = \boldsymbol{0}$, $\boldsymbol{d}_a = \text{diag}\,\{1, 1, \sigma, \sigma, 0\}$

$$\boldsymbol{\Gamma} = \begin{pmatrix} (-u_{1x}/\kappa^2, -u_{1y}/\kappa^2)^T \\ (-u_{2x}/\kappa^2, -u_{2y}/\kappa^2)^T \\ (0, u_{4x} - u_{3y} - B_{az})^T \\ (-u_{4x} + u_{3y} + B_{az}, 0)^T \\ (u3, u4)^T \end{pmatrix}, \quad \boldsymbol{F} = \begin{pmatrix} F_1 \\ F_2 \\ (u_1 u_{2x} - u_2 u_{1x})/\kappa - (u_1^2 + u_2^2)u_3 \\ (u_1 u_{2y} - u_2 u_{1y})/\kappa - (u_1^2 + u_2^2)u_3 \\ u_{3x} + u_{4y} + u_5 \end{pmatrix}$$

Zero flux

$$-\boldsymbol{n} \cdot \boldsymbol{\Gamma} = 0, \text{ on } \Omega$$

TDGLE in COMSOL

$$\boldsymbol{e}_a \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \boldsymbol{d}_a \frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{\Gamma} = \boldsymbol{F}$$

$\boldsymbol{e}_a = \boldsymbol{0}$, $\boldsymbol{d}_a = \text{diag}\{1, 1, \sigma, \sigma, 0\}$

$$\boldsymbol{\Gamma} = \begin{pmatrix} (-u_{1x}/\kappa^2, -u_{1y}/\kappa^2)^T \\ (-u_{2x}/\kappa^2, -u_{2y}/\kappa^2)^T \\ (0, u_{4x} - u_{3y} - B_{az})^T \\ (-u_{4x} + u_{3y} + B_{az}, 0)^T \\ (u3, u4)^T \end{pmatrix}, \quad \boldsymbol{F} = \begin{pmatrix} F_1 \\ F_2 \\ (u_1 u_{2x} - u_2 u_{1x})/\kappa - (u_1^2 + u_2^2)u_3 \\ (u_1 u_{2y} - u_2 u_{1y})/\kappa - (u_1^2 + u_2^2)u_3 \\ u_{3x} + u_{4y} + u_5 \end{pmatrix}$$

Zero flux

$$-\boldsymbol{n} \cdot \boldsymbol{\Gamma} = 0, \text{ on } \Omega$$

Initial conditions

$$u_1 = 1, \quad u_2 = 0, \quad u_3 = 0, \quad u_4 = 0, \quad u_5 = 0$$

PDE simulation flowchart

1. Model initialization
2. Global and local definitions
3. Geometry settings
4. PDE interface settings
5. Mesh generation
6. PDE solution
7. Postprocessing

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5. Mesh generation
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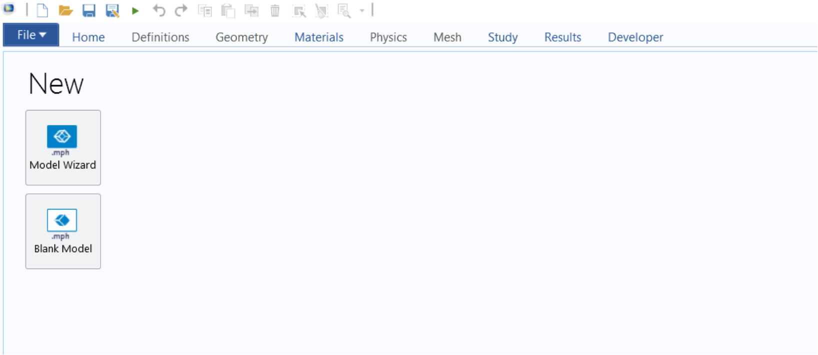
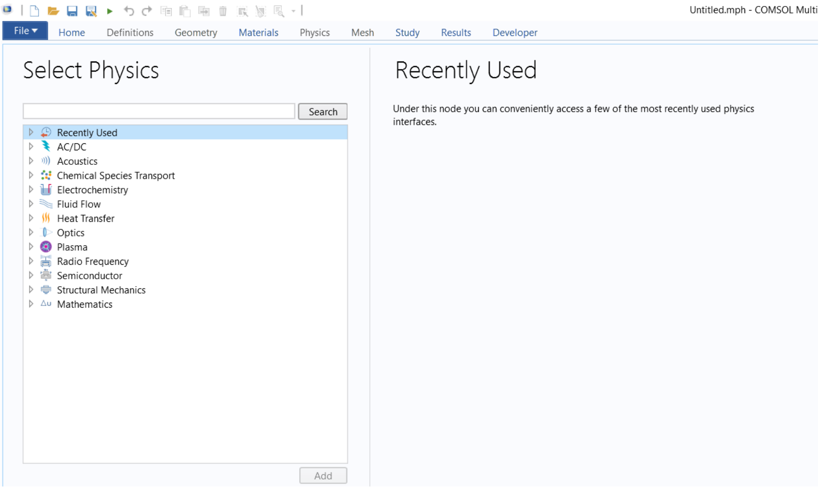
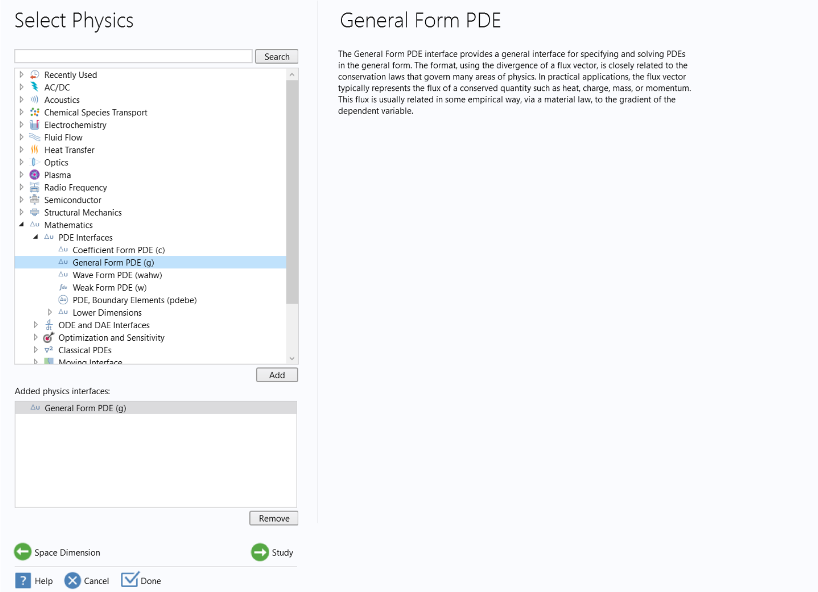
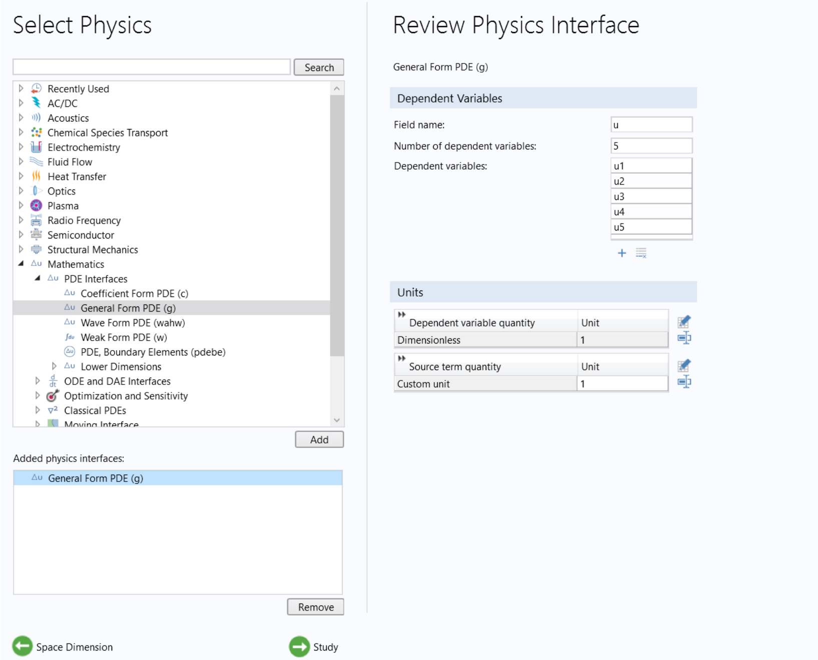
7. Postprocessing

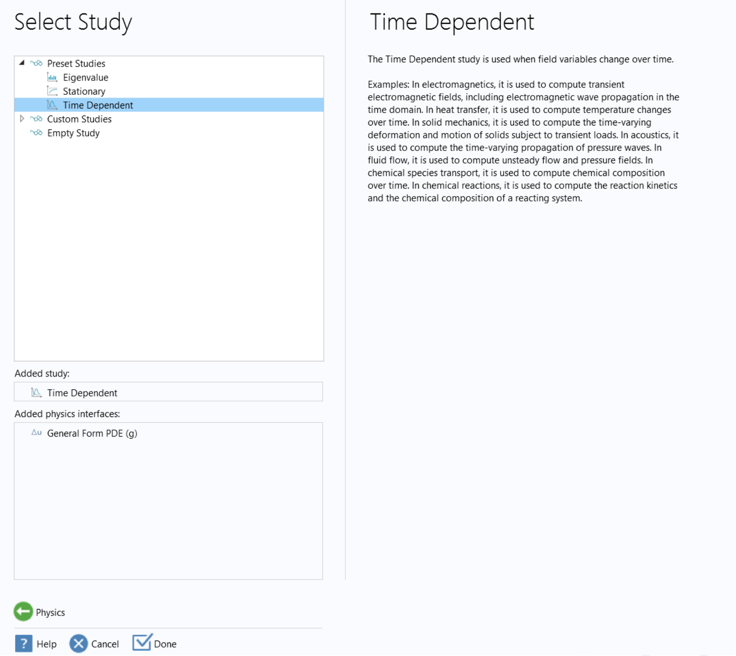
Model initialization



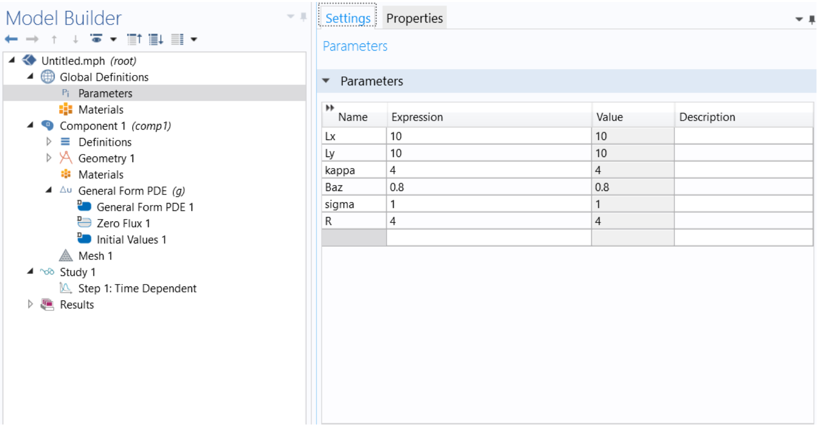

Untitled.mph - COMSOL Multi
File
Home
Definitions
Geometry
Materials
Physics
Mesh
Study
Results
Developer
Select Physics
Search
Recently Used
AC/DC
Acoustics
Chemical Species Transport
Electrochemistry
Fluid Flow
Heat Transfer
Optics
Plasma
Radio Frequency
Semiconductor
Structural Mechanics
Mathematics
Add
Recently Used
Under this node you can conveniently access a few of the most recently used physics interfaces.


Select Physics
Search
Recently Used
AC/DC
Acoustics
Chemical Species Transport
Electrochemistry
Fluid Flow
Heat Transfer
Optics
Plasma
Radio Frequency
Semiconductor
Structural Mechanics
Mathematics
PDE Interfaces
Coefficient Form PDE (c)
General Form PDE (g)
Wave Form PDE (wahw)
Weak Form PDE (w)
PDE, Boundary Elements (pdebe)
Lower Dimensions
ODE and DAE Interfaces
Optimization and Sensitivity
Classical PDEs
Moving Interface
Add
Added physics interfaces:
General Form PDE (g)
Remove
Space Dimension
Study
Help
Cancel
Done
General Form PDE
The General Form PDE interface provides a general interface for specifying and solving PDEs in the general form. The format, using the divergence of a flux vector, is closely related to the conservation laws that govern many areas of physics. In practical applications, the flux vector typically represents the flux of a conserved quantity such as heat, charge, mass, or momentum. This flux is usually related in some empirical way, via a material law, to the gradient of the dependent variable.

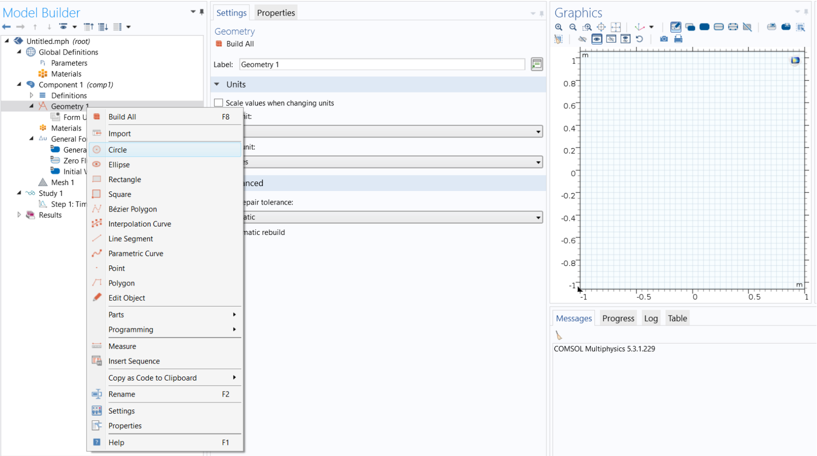

Select Physics
Search
Recently Used
AC/DC
Acoustics
Chemical Species Transport
Electrochemistry
Fluid Flow
Heat Transfer
Optics
Plasma
Radio Frequency
Semiconductor
Structural Mechanics
Mathematics
PDE Interfaces
Coefficient Form PDE (c)
General Form PDE (g)
Wave Form PDE (wahw)
Weak Form PDE (w)
PDE, Boundary Elements (pdebe)
Lower Dimensions
ODE and DAE Interfaces
Optimization and Sensitivity
Classical PDEs
Add
Added physics interfaces:
General Form PDE (g)
Remove
Space Dimension
Study
Review Physics Interface
General Form PDE (g)
Dependent Variables
Field name: u
Number of dependent variables: 5
Dependent variables: u1 u2 u3 u4 u5
Units
Dependent variable quantity | Unit
Dimensionless | 1
Source term quantity | Unit
Custom unit | 1

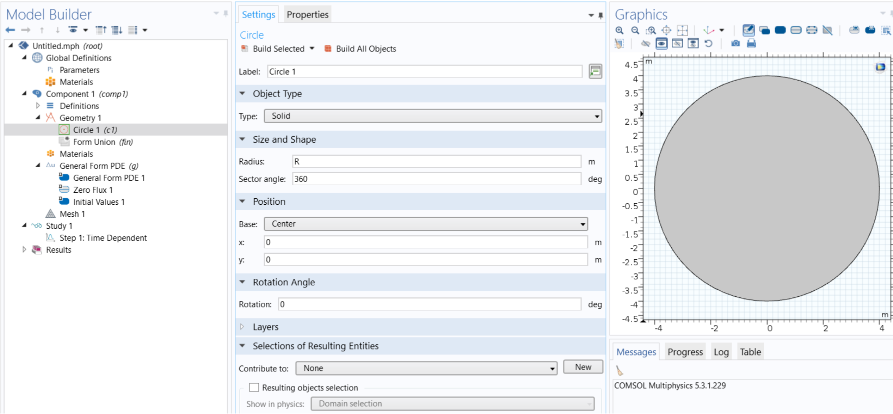

Select Study
Preset Studies
Eigenvalue
Stationary
Time Dependent
Custom Studies
Empty Study
Added study:
Time Dependent
Added physics interfaces:
General Form PDE (g)
Physics
Help
Cancel
Done
Time Dependent
The Time Dependent study is used when field variables change over time.
Examples: In electromagnetics, it is used to compute transient electromagnetic fields, including electromagnetic wave propagation in the time domain. In heat transfer, it is used to compute temperature changes over time. In solid mechanics, it is used to compute the time-varying deformation and motion of solids subject to transient loads. In acoustics, it is used to compute the time-varying propagation of pressure waves. In fluid flow, it is used to compute unsteady flow and pressure fields. In chemical species transport, it is used to compute chemical composition over time. In chemical reactions, it is used to compute the reaction kinetics and the chemical composition of a reacting system.

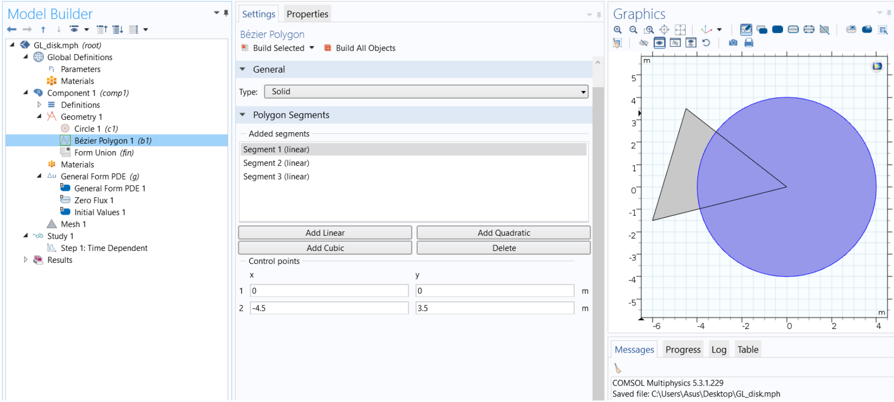
Global and local definitions

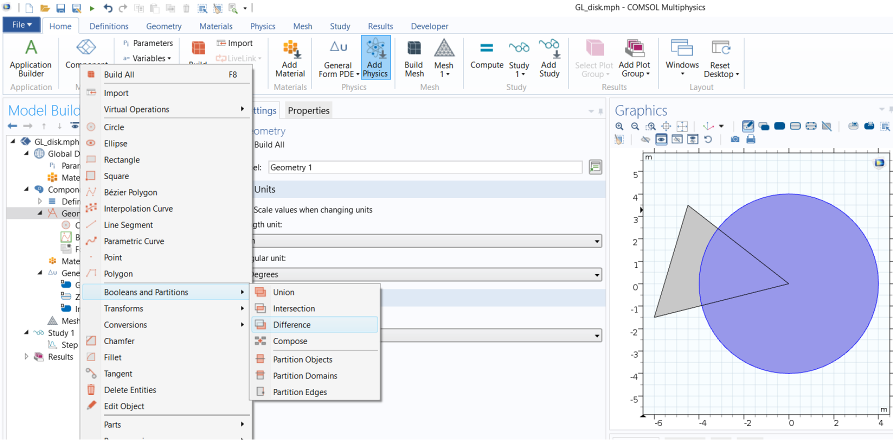


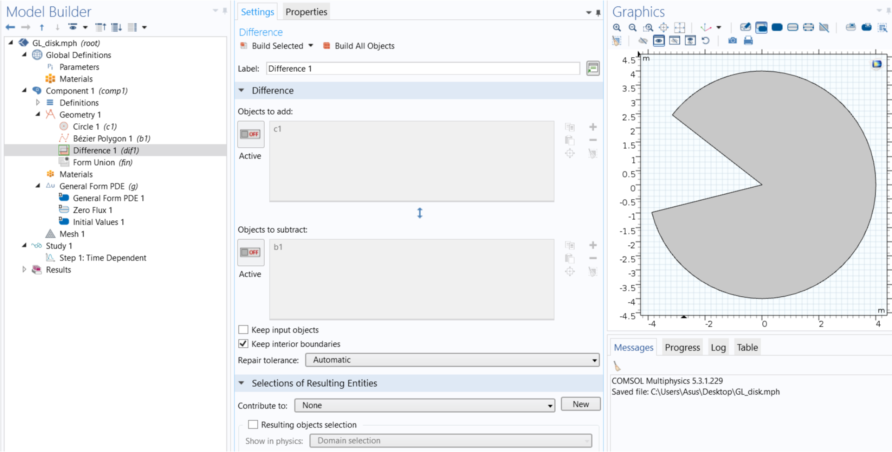
Geometry settings



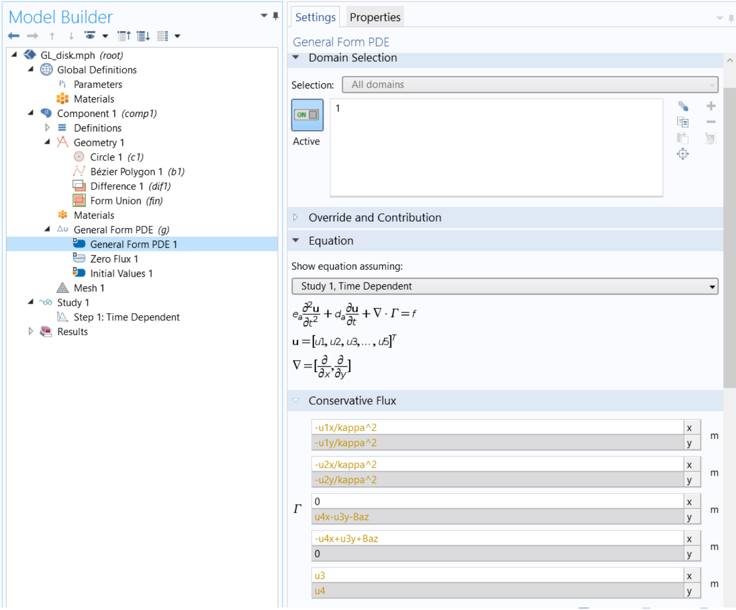

Model Builder
Untitled.mph (root)
Global Definitions
Parameters
Materials
Component 1 (comp1)
Definitions
Geometry 1
Circle 1 (c1)
Form Union (fin)
Materials
General Form PDE (g)
General Form PDE 1
Zero Flux 1
Initial Values 1
Mesh 1
Study 1
Step 1: Time Dependent
Results
Settings
Properties
Circle
Build Selected
Build All Objects
Label: Circle 1
Object Type
Type: Solid
Size and Shape
Radius: R m
Sector angle: 360 deg
Position
Base: Center
x: 0 m
y: 0 m
Rotation Angle
Rotation: 0 deg
Layers
Selections of Resulting Entities
Contribute to: None
New
Resulting objects selection
Show in physics: Domain selection
Graphics
Messages
Progress
Log
Table
COMSOL Multiphysics 5.3.1.229


Model Builder
GL_disk.mph (root)
Global Definitions
Parameters
Materials
Component 1 (comp1)
Definitions
Geometry 1
Circle 1 (c1)
Bézier Polygon 1 (b1)
Form Union (fin)
Materials
General Form PDE (g)
General Form PDE 1
Zero Flux 1
Initial Values 1
Mesh 1
Study 1
Step 1: Time Dependent
Results
Settings
Properties
Bézier Polygon
Build Selected
Build All Objects
General
Type: Solid
Polygon Segments
Added segments
Segment 1 (linear)
Segment 2 (linear)
Segment 3 (linear)
Add Linear
Add Quadratic
Add Cubic
Delete
Control points
x
y
1 0 0 m
2 -4.5 3.5 m
Graphics
Messages
Progress
Log
Table
COMSOL Multiphysics 5.3.1.229
Saved file: C:\Users\Asus\Desktop\GL_disk.mph


GL_disk.mph - COMSOL Multiphysics
File
Home
Definitions
Geometry
Materials
Physics
Mesh
Study
Results
Developer
Application Builder
Parameters
Variables
Import
LiveLink
Add Material
General Form PDE
Add Physics
Build Mesh
Mesh 1
Compute
Study 1
Add Study
Select Plot Group
Add Plot Group
Windows
Reset Desktop
Model Builder
Build All F8
Import
Virtual Operations
Circle
Ellipse
Rectangle
Square
Bézier Polygon
Interpolation Curve
Line Segment
Parametric Curve
Point
Polygon
Booleans and Partitions
Transforms
Conversions
Chamfer
Fillet
Tangent
Delete Entities
Edit Object
Parts
Union
Intersection
Difference
Compose
Partition Objects
Partition Domains
Partition Edges
Properties
Geometry 1
Units
Scale values when changing units
Degrees
Graphics


Model Builder
GL_disk.mph (root)
Global Definitions
Parameters
Materials
Component 1 (comp1)
Definitions
Geometry 1
Circle 1 (c1)
Bézier Polygon 1 (b1)
Difference 1 (dif1)
Form Union (fin)
Materials
General Form PDE (g)
General Form PDE 1
Zero Flux 1
Initial Values 1
Mesh 1
Study 1
Step 1: Time Dependent
Results
Settings
Properties
Difference
Build Selected
Build All Objects
Label: Difference 1
Difference
Objects to add:
c1
Active
Objects to subtract:
b1
Active
Keep input objects
Keep interior boundaries
Repair tolerance: Automatic
Selections of Resulting Entities
Contribute to: None
New
Resulting objects selection
Show in physics: Domain selection
Graphics
Messages
Progress
Log
Table
COMSOL Multiphysics 5.3.1.229
Saved file: C:\Users\Asus\Desktop\GL_disk.mph

PDE interface settings



Damping or Mass Coefficient

	1	s	0	s	0	s	0	s	0	s
	0	s	1	s	0	s	0	s	0	s
d_a	0	s	0	s	sigma	s	0	s	0	s
	0	s	0	s	0	s	sigma	s	0	s
	0	s	0	s	0	s	0	s	0	s

Mass Coefficient

	0	s^2	0	s^2	0	s^2	0	s^2	0	s^2
	0	s^2	0	s^2	0	s^2	0	s^2	0	s^2
e_a	0	s^2	0	s^2	0	s^2	0	s^2	0	s^2
	0	s^2	0	s^2	0	s^2	0	s^2	0	s^2
	0	s^2	0	s^2	0	s^2	0	s^2	0	s^2

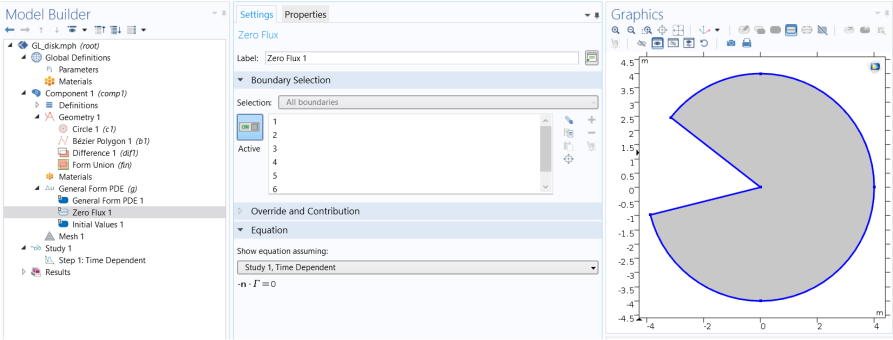
Conservative Flux

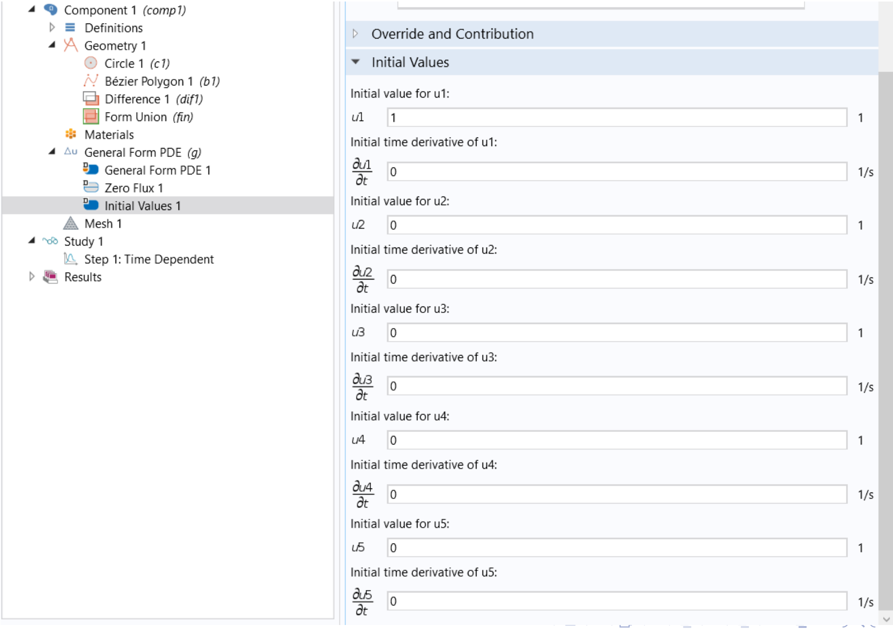
Source Term

f

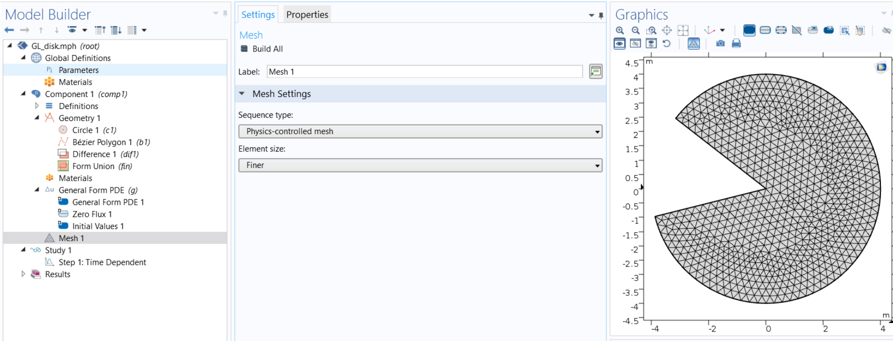
```
(u3x+u4y)*u2/kappa+2*(u3*u2x+u4*u2y)/kappa-(u3^2+u4^2)*u1+u1-(u1^2+u2^2)*u1
-(u3x+u4y)*u1/kappa-2*(u3*u1x+u4*u1y)/kappa-(u3^2+u4^2)*u2+u2-(u1^2+u2^2)*u2
(u1*u2x-u2*u1x)/kappa-(u1^2+u2^2)*u3
(u1*u2y-u2*u1y)/kappa-(u1^2+u2^2)*u4
u3x+u4y+u5
```

Damping or Mass Coefficient

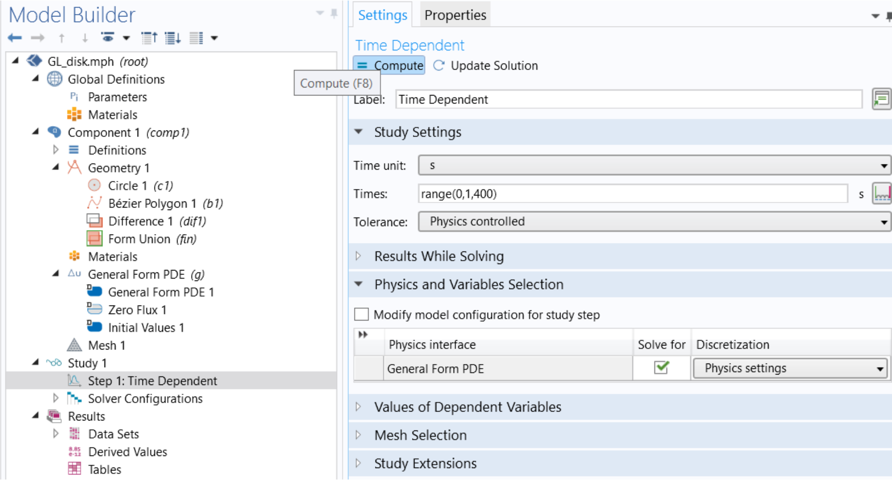

Model Builder
GL_disk.mph (root)
Global Definitions
Parameters
Materials
Component 1 (comp1)
Definitions
Geometry 1
Circle 1 (c1)
Bézier Polygon 1 (b1)
Difference 1 (dif1)
Form Union (fin)
Materials
General Form PDE (g)
General Form PDE 1
Zero Flux 1
Initial Values 1
Mesh 1
Study 1
Step 1: Time Dependent
Results
Settings
Properties
Zero Flux
Label: Zero Flux 1
Boundary Selection
Selection: All boundaries
1
2
3
4
5
6
Active
Override and Contribution
Equation
Show equation assuming:
Study 1, Time Dependent
-n · Γ = 0
Graphics
m
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-4
-2
0
2
4

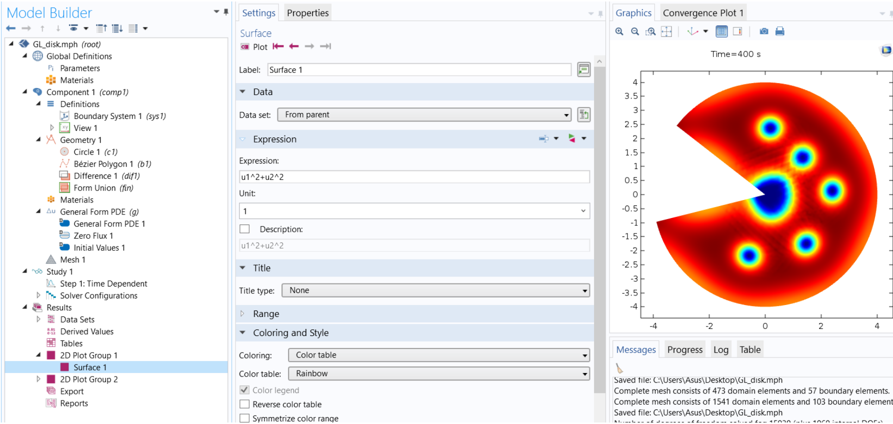

Component 1 (comp1)
Definitions
Geometry 1
Circle 1 (c1)
Bézier Polygon 1 (b1)
Difference 1 (dif1)
Form Union (fin)
Materials
General Form PDE (g)
General Form PDE 1
Zero Flux 1
Initial Values 1
Mesh 1
Study 1
Step 1: Time Dependent
Results
Override and Contribution
Initial Values
Initial value for u1:
u1 1 1
Initial time derivative of u1:
∂u1/∂t 0 1/s
Initial value for u2:
u2 0 1
Initial time derivative of u2:
∂u2/∂t 0 1/s
Initial value for u3:
u3 0 1
Initial time derivative of u3:
∂u3/∂t 0 1/s
Initial value for u4:
u4 0 1
Initial time derivative of u4:
∂u4/∂t 0 1/s
Initial value for u5:
u5 0 1
Initial time derivative of u5:
∂u5/∂t 0 1/s

Mesh generation



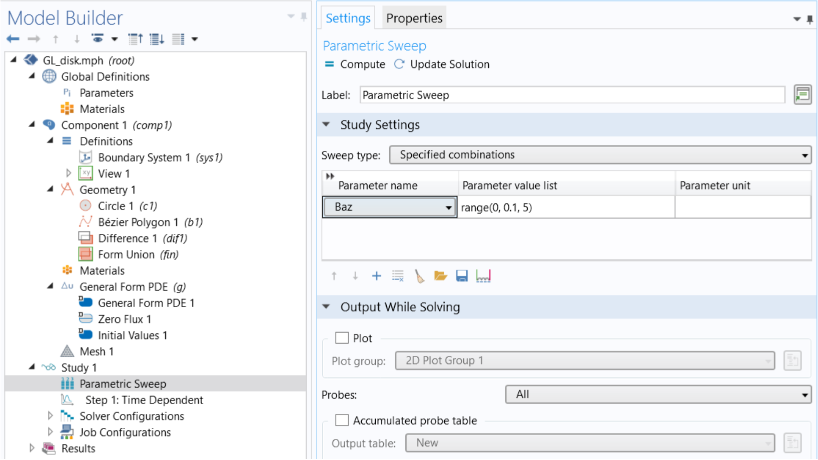
PDE solution




Model Builder
GL_disk.mph (root)
Global Definitions
Parameters
Materials
Component 1 (comp1)
Definitions
Boundary System 1 (sys1)
View 1
Geometry 1
Circle 1 (c1)
Bézier Polygon 1 (b1)
Difference 1 (dif1)
Form Union (fin)
Materials
General Form PDE (g)
General Form PDE 1
Zero Flux 1
Initial Values 1
Mesh 1
Study 1
Step 1: Time Dependent
Solver Configurations
Results
Data Sets
Derived Values
Tables
2D Plot Group 1
Surface 1
2D Plot Group 2
Export
Reports
Settings
Properties
Surface
Plot
Label: Surface 1
Data
Data set: From parent
Expression
Expression:
u1^2+u2^2
Unit:
1
Description:
u1^2+u2^2
Title
Title type: None
Range
Coloring and Style
Coloring: Color table
Color table: Rainbow
Color legend
Reverse color table
Symmetrize color range
Graphics
Convergence Plot 1
Time=400 s
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4
-2
0
2
4
Messages
Progress
Log
Table
Saved file: C:\Users\Asus\Desktop\GL_disk.mph
Complete mesh consists of 473 domain elements and 57 boundary elements.
Complete mesh consists of 1541 domain elements and 103 boundary element
Saved file: C:\Users\Asus\Desktop\GL_disk.mph

Cooper pair density $|\psi|^2$, $B_{az} = 0.8$

Parametric sweep



Cooper pair density $|\psi|^2$ dependence on B_a

Thank you for attention!

The time dependent Ginzburg-Landau (TDGL) equation

$$\frac{\hbar^2}{2mD}\left(\frac{\partial}{\partial t}+i\frac{q}{\hbar}\Phi\right)\Psi=-\frac{1}{2m}\left(\frac{\hbar}{i}\nabla-q\boldsymbol{A}\right)^2\Psi+\alpha\Psi-\beta\left|\Psi\right|^2\Psi$$

$$\sigma\left(\frac{\partial\boldsymbol{A}}{\partial t}+\nabla\Phi\right)=\frac{q\hbar}{2mi}\left(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*\right)-\frac{q^2}{m}\left|\Psi\right|^2\boldsymbol{A}-\frac{1}{\mu_0}\nabla\times(\nabla\times\boldsymbol{A}-\boldsymbol{B}_a)$$

The time dependent Ginzburg-Landau (TDGL) equation

$$\frac{\hbar^2}{2mD}\left(\frac{\partial}{\partial t}+i\frac{q}{\hbar}\Phi\right)\Psi=-\frac{1}{2m}\left(\frac{\hbar}{i}\nabla-q\boldsymbol{A}\right)^2\Psi+\alpha\Psi-\beta\left|\Psi\right|^2\Psi$$

$$\sigma\left(\frac{\partial \boldsymbol{A}}{\partial t}+\nabla\Phi\right)=\frac{q\hbar}{2mi}\left(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*\right)-\frac{q^2}{m}\left|\Psi\right|^2\boldsymbol{A}-\frac{1}{\mu_0}\nabla\times(\nabla\times\boldsymbol{A}-\boldsymbol{B}_a)$$

Boundary conditions

$$\left(\frac{\hbar}{i}\nabla\Psi-q\boldsymbol{A}\Psi\right)\cdot\boldsymbol{n}=0,\quad \text{on } \partial\Omega$$

$$\boldsymbol{B}_i=\boldsymbol{B}_a,\quad \text{on } \partial\Omega$$

$$\left(\frac{\partial \boldsymbol{A}}{\partial t}+\nabla\Phi\right)\cdot\boldsymbol{n}=0,\quad \text{on } \partial\Omega$$

T.S. Alstrøm et al. Acta Appl. Math 115 (2011) 63

Normalization

$$(x, y, z, t) = \left(\lambda x', \lambda y', \lambda z', \frac{\xi^2}{D} t'\right), \quad \boldsymbol{A} = \frac{\hbar}{q\xi} \boldsymbol{A}'$$

$$\Psi = \sqrt{\frac{\alpha}{\beta}} \Psi', \quad \Phi = \alpha D \kappa^2 \sqrt{\frac{2\mu_0}{b}} \Phi', \quad \sigma = \frac{1}{\mu_0 D \kappa^2} \sigma'$$

$$\left(\frac{\partial}{\partial t} + i\kappa\Phi\right)\Psi = -\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \Psi + \Psi - |\Psi|^2 \Psi$$

$$\sigma\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right) = \frac{1}{2i\kappa}\left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*\right) - |\Psi|^2 \boldsymbol{A} - \left(\nabla \times \boldsymbol{A} - \boldsymbol{B}_a\right)$$

Normalization

$$(x, y, z, t) = \left(\lambda x', \lambda y', \lambda z', \frac{\xi^2}{D} t'\right), \quad \boldsymbol{A} = \frac{\hbar}{q\xi} \boldsymbol{A}'$$

$$\Psi = \sqrt{\frac{\alpha}{\beta}} \Psi', \quad \Phi = \alpha D \kappa^2 \sqrt{\frac{2\mu_0}{b}} \Phi', \quad \sigma = \frac{1}{\mu_0 D \kappa^2} \sigma'$$

$$\left(\frac{\partial}{\partial t} + i\kappa\Phi\right) \Psi = -\left(\frac{i}{\kappa} \nabla + \boldsymbol{A}\right)^2 \Psi + \Psi - |\Psi|^2 \Psi$$

$$\sigma \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right) = \frac{1}{2i\kappa} \left(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*\right) - |\Psi|^2 \boldsymbol{A} - (\nabla \times \boldsymbol{A} - \boldsymbol{B}_a)$$

Boundary conditions

$$\left(\frac{\hbar}{i} \nabla \Psi + \boldsymbol{A}\Psi\right) \cdot \boldsymbol{n} = 0, \quad \text{on } \partial\Omega$$

$$\boldsymbol{B}_i = \boldsymbol{B}_a, \quad \text{on } \partial\Omega$$

$$\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right) \cdot \boldsymbol{n} = 0, \quad \text{on } \partial\Omega$$

Gauge invariance

$$\tilde{\Psi} = \Psi e^{i\kappa\chi}, \quad \tilde{\boldsymbol{A}} = \boldsymbol{A} + \nabla\chi, \quad \tilde{\Phi} = \Phi - \frac{\partial\chi}{\partial t}$$

Zero electric field potential gauge

$$\tilde{\Phi} = 0$$

$$\frac{\partial\chi}{\partial t} = \Phi$$