A general picture

Association: the ray $\lambda_0 = (\alpha, 0))$ eigenstate of Z^α

$$Z^\alpha |0\rangle = |0\rangle,$$

Association: the states $|n\rangle = X^n|0\rangle \Leftrightarrow$ the parallel lines $\beta = n$

In general

$$\begin{aligned} \lambda_m \quad &: \quad \beta = m\alpha \Leftrightarrow V^m|0\rangle = |\psi_m^0\rangle \\ |\psi_m^n\rangle \quad &= \quad X^n|\psi_m^0\rangle \Leftrightarrow \beta = m\alpha + n \end{aligned}$$

The Wigner function of the state $|\psi_m^n\rangle$

Using the covariance property of the Wigner function

$$W_{|\psi_0^n\rangle\langle\psi_0^n|}(\alpha,\beta) = W_{X^n|0\rangle\langle 0|(X^\dagger)^n}(\alpha,\beta) = W_{|0\rangle\langle 0|}(\alpha,\beta - n) = \frac{1}{p}\delta_{\beta,n},$$

and

$$\hat{w}(\alpha,\beta) = \frac{1}{p}\sum_{\gamma,\delta=0}^{p-1} \omega(\alpha\delta - \beta\gamma)\phi(\gamma,\delta) Z^\gamma X^\delta,$$

$$\begin{aligned} W_{|\psi_m^n\rangle\langle\psi_m^n|}(\alpha,\beta) &= \frac{1}{p} Tr(\hat{w}(\alpha,\beta) V^m X^n |0\rangle\langle 0| X^{\dagger n} V^{\dagger m}) = \\ &= \frac{1}{p} Tr(X^{\dagger n} V^{\dagger m} \hat{w}(\alpha,\beta) V^m X^n |0\rangle\langle 0|) \\ &= \frac{1}{p^2}\sum_{\gamma,\delta=0}^{p-1} \omega(\alpha\delta - \beta\gamma)\omega(-n\gamma)\phi(\gamma,\delta)\omega(2^{-1}m\gamma^2)\langle 0|Z^\gamma X^{\delta - m\gamma}|0\rangle \\ &= \frac{1}{p}\delta_{\beta, m\alpha+n}. \end{aligned}$$

The Wigner function of $|\psi_m^n\rangle$ has the form of the line $\beta = m\alpha + n$.

0.1 Mutually unbiased bases

The inner product of eigenstates

$$|\langle\psi_{m'}^{n'}|\psi_m^n\rangle| = \langle 0|(V^\dagger)^m (X^\dagger)^n X^{n'} V^{m'}|0\rangle \tag{1}$$

as V and X commute,

$$V = \sum_{n=0}^{p-1} c_n |\widetilde{n}\rangle\langle\widetilde{n}|, \langle n|\tilde{k}\rangle = \omega(kn)/\sqrt{p}$$

$$\langle 0|X^{n'-n}V^{m'-m}|0\rangle = \sum_{k=0}^{p-1} c_k^{m'-m}\langle n-n'|\tilde{k}\rangle\langle\tilde{k}|0\rangle = \frac{1}{p}\Psi^{n,n'}_{m,m'},$$

where

$$\Psi^{n,n'}_{m,m'} \equiv \sum_{k=0}^{p-1} c_k^{m'-m}\omega(k(n-n')).$$

Then

$$|\Psi^{n,n'}_{m,m'}|^2 = \sum_{k,k'=0}^{p-1} c_k^{m'-m} c_{k'}^{*m'-m}\omega((k-k')(n-n')),$$

changing the index $k-k'=l$, the above formula is rewritten as

$$\sum_{k',l=0}^{p-1} c_{k'+l}^{m'-m} c_{k'}^{*m'-m}\omega(l(n-n')),$$

using the relation

$$\begin{aligned} c_{n+\alpha}c_n^* &= \varphi\omega(-n\alpha) \\ c_\alpha &= \varphi = \omega(-2^{-1}\alpha^2) \quad \text{for odd } p \end{aligned}$$

one obtains

$$\begin{aligned} &\sum_{k',l=0}^{p-1} \varphi^{m-m'}(l)\omega(-lk'(m-m'))\omega(l(n-n')) \\ = &\left\{ \begin{array}{cc} p^2\delta_{n,n'} & m=m' \\ p\sum_{l=0}^{p-1}\varphi^{m-m'}(l)\omega(l(n-n'))\delta_{l,0} & m\neq m' \end{array} \right. , \end{aligned}$$

giving the two possible results

$$\frac{1}{p^2}|\Psi^{n,n'}_{m,m'}|^2 = \left\{ \begin{array}{cc} \delta_{n,n'} & m=m' \\ 1/p & m\neq m' \end{array} \right.$$

When the inner product of basis elements has the value

$$|\langle\psi^{n'}_{m'}|\psi^n_m\rangle| = \left\{ \begin{array}{cc} \delta_{n,n'} & m=m' \\ \frac{1}{\sqrt{p}} & m\neq m' \end{array} \right. , \tag{2}$$

the bases are *mutually unbiased* or complementary.

For each m-striation

$$\sum_{n=0}^{p-1} |\psi^n_m\rangle\langle\psi^n_m| = \hat{I}$$

and the orthogonality relation

$$Tr\left[\left(|\psi^n_m\rangle\langle\psi^n_m| - \frac{1}{p}\hat{I}\right)\left(|\psi^{n'}_{m'}\rangle\langle\psi^{n'}_{m'}| - \frac{1}{p}\hat{I}\right)\right] = 0$$

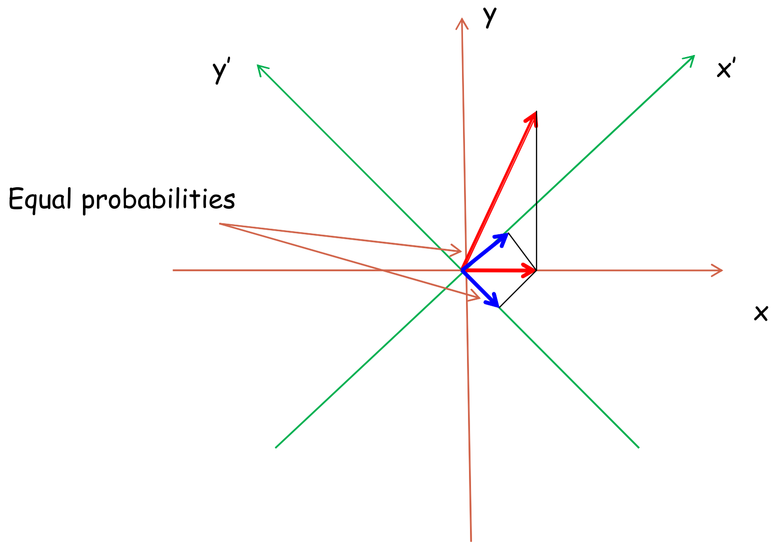


Figure 1:

Recall that $|\psi_m^n\rangle$ are eigenstates of the commuting sets $\{D(\alpha, m\alpha), \alpha = 1, ..., p-1)\}$

Observe that the operator sets corresponding to rays $\beta = m\alpha$, $\alpha = 0$ are disjoint,

$$Tr\left(D(\alpha, m\alpha)D(\alpha', m'\alpha')\right) = 0$$

Theorem: Eigenstates of the mutually disjoint sets are mutually unbiased.

Reconstruction procedure

Let us calculate the average of the density matrix on the state corresponding to the line $\beta = m\alpha + n$

$$\langle\psi_m^n|\rho|\psi_m^n\rangle = \langle 0|(V^\dagger)^m(X^\dagger)^n\rho X^n V^m|0\rangle, \tag{3}$$

using the expansion

$$\begin{aligned} \rho &= \sum_{\alpha,\beta=0}^{p-1} W_\rho(\alpha,\beta)\hat{w}(\alpha,\beta), \\ \hat{w}(\alpha,\beta) &= \frac{1}{p}\sum_{\gamma,\delta=0}^{p-1}\omega(\alpha\delta-\beta\gamma)\underbrace{\omega(-2^{-1}\gamma\delta)Z^\gamma X^\delta}_{D(\gamma,\delta)} \end{aligned}$$

and the rotation operator action

$$V^m Z^\gamma (V^\dagger)^m = \omega(-2^{-1} m\gamma^2) Z^\gamma X^{m\gamma}$$

we rewrite the average (3) for odd prime case as

$$\begin{aligned}\langle\psi_m^n|\rho|\psi_m^n\rangle = \quad & \frac{1}{p}\sum_{\alpha,\beta} W(\alpha,\beta)\sum_{\gamma,\delta}\omega(\alpha\delta-\beta\gamma)\omega(-2^{-1}\gamma\delta)\\ & \times\omega(-2^{-1}m\gamma^2)\langle n|Z^\gamma X^{\delta-m\gamma}|n\rangle,\end{aligned}$$

Taking into account that

$$\langle n|Z^\gamma X^{\delta-m\gamma}|n\rangle = \omega\left(\gamma n\right)\delta_{m\gamma,\delta}$$

we get

$$\langle\psi_m^n|\rho|\psi_m^n\rangle = \sum_{\alpha,\beta=0}^{p-1} W(\alpha,\beta)\delta_{\beta,m\alpha+n},$$

which is a *tomographic condition.*

For the line $\alpha = 0$,

$$\begin{aligned}\langle\tilde{l}|\rho|\tilde{l}\rangle &= \langle l|F^\dagger\rho F|l\rangle\\ &= \frac{1}{p}\sum_{\alpha,\beta=0}^{p-1} W(\alpha,\beta)\sum_{\gamma,\delta=0}^{p-1}\omega(\alpha\delta-\beta\gamma)\phi(\gamma,\delta)\langle l|X^\gamma Z^{-\delta}|l\rangle,\end{aligned}$$

so that

$$\langle\tilde{l}|\rho|\tilde{l}\rangle = \sum_{\alpha,\beta=0}^{p-1} W(\alpha,\beta)\delta_{\alpha,l}.$$

The sum of the Wigner function along a line gives the probability of the system to be found in the state associated to this line.

Restoring an operator from its tomogram

The tomogram $\omega(m,n) = \langle\psi_m^n|\rho|\psi_n^n\rangle$ with the Wigner function:

$$\omega(m,n) = \sum_{\alpha,\beta} W(\alpha,\beta)\delta_{\beta,m\alpha+n} = \sum_\alpha W(\alpha,m\alpha+n), \tag{4}$$

The elements of the density matrix in the basis of the displacement operators $\rho_{\kappa,\mu\kappa}$,

$$\omega(m,n) = \sum_\kappa \rho_{\kappa,m\kappa}\chi(\kappa n).$$

this can be immediately inverted

$$\rho_{\kappa,m\kappa} = \sum_n \omega(mn,n)\chi(-\kappa n),$$

or, changing the indices

$$\rho_{\kappa,\lambda} = \sum_n \omega(\kappa^{-1}\lambda, n)\chi(-\kappa n),$$

where $\kappa \neq 0$.

To reconstruct the matrix element $\rho_{0,\lambda}$ we have to use the conjugate basis $|\widetilde{\kappa}\rangle$:

$$\omega\,(\kappa) = \langle\widetilde{\kappa}|\rho|\widetilde{\kappa}\rangle = \sum_{\alpha,\beta} W(\alpha,\beta)\delta_{\alpha,\kappa} = \sum_{\lambda} \rho_{0,\lambda}\chi(-\kappa\lambda),$$

which leads to

$$\rho_{0,\lambda} = \sum_{\kappa} \omega(\kappa)\chi(\kappa\lambda).$$

Unbisedness relation

$$|\langle\psi_{m'}^{n'}|\psi_m^n\rangle|^2 = \delta_{nn'}\delta_{mm'} + \frac{1}{p}(1-\delta_{mm'}),$$

Wigner kernel in terms of unbiased projectors

$$\rho = \sum_{\alpha,\beta=0}^{p-1} W_\rho(\alpha,\beta)\hat{w}(\alpha,\beta), \quad W_\rho(\alpha,\beta) = \frac{1}{p}\mathrm{Tr}\,(\rho\hat{w}(\alpha,\beta))$$

$$\begin{aligned}
\hat{w}(\alpha,\beta) &= \sum_{m,n} |\psi_m^n\rangle\langle\psi_m^n|\delta_{\beta,m\alpha+n} + \sum_n |\tilde{n}\rangle\langle\tilde{n}|\delta_{\alpha,n} - \hat{I} \\
W_\rho(\alpha,\beta) &= \frac{1}{p}\left[\sum_k \underbrace{\langle\psi_{(\alpha,\beta)}^{(k)}|\rho|\psi_{(\alpha,\beta)}^{(k)}\rangle}_{\text{projections into lines crossing }(\alpha,\beta)} - 1\right]
\end{aligned}$$

Wigner function of a state $|\psi_m^n\rangle$

$$W_{|\psi_m^n\rangle\langle\psi_m^n|}\,(\alpha,\beta) = \frac{1}{p}\delta_{\beta,m\alpha+n}$$

Theorem: The Wigner function is positive only for states $|\psi_m^n\rangle, m,n = 0,...,p-1$ and $|\tilde{n}\rangle, n = 0,...,p-1$

Density matrix reconstruction

$$\begin{aligned}
\rho &= \sum_{m,n} |\psi_m^n\rangle\langle\psi_m^n|p_{mn} + \sum_n \tilde{p}_n|\tilde{n}\rangle\langle\tilde{n}| - \hat{I} \\
p_{mn} &= \langle\psi_m^n|\rho|\psi_m^n\rangle, \quad \tilde{p}_n = \langle\tilde{n}|\rho|\tilde{n}\rangle, \\
\sum_{n=0}^{p-1} p_{mn} &= 1, \quad \sum_{n=0}^{p-1} \tilde{p}_n = 1
\end{aligned}$$

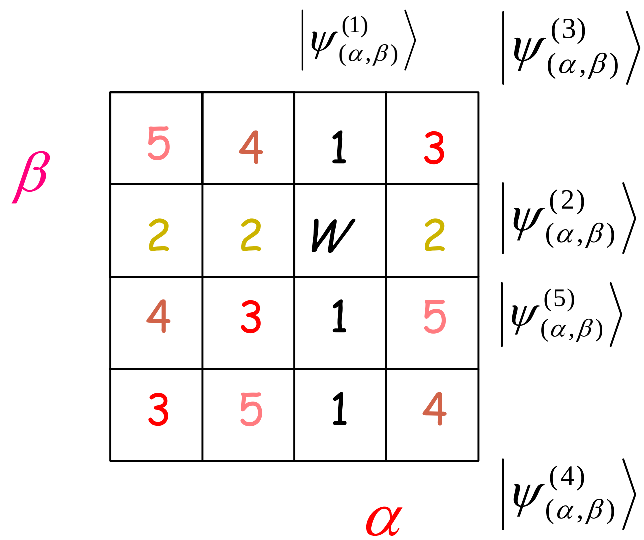


Figure 2:

For a reonstruction of $\acute{p} \times p$ matrix one needs $p+1$ mutually unbiased sets, each containing $p-1$ states.

Such sets are common eigenstates of p commuting operators

$$
\begin{array}{l}
Z,\ Z^2, ..., Z^{p-1} \\
X,\ X^2, ..., X^{p-1} \\
ZX,\ (ZX)^2, ..., (ZX)^{p-1} \\
ZX^2,\ (ZX^2)^2, ..., (ZX^2)^{p-1} \\
... \\
.. \\
ZX^{p-1},\ (ZX^{p-1})^2, ..., (ZX^{p-1})^{p-1}
\end{array}
$$

The sets of commuting operators are mutually disjoint

$$Tr\left[(ZX^n)^m, (ZX^{n'})^{m'}\right] = 0, \quad m \neq m'.$$

Geometrically unbiased bases are described by (non-intersecting) rays in the discrete phase-space.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 6 & 4 & 3 & 7 & 2 & 5 \\ 2 & 6 & 0 & 7 & 5 & 4 & 1 & 3 \\ 3 & 4 & 7 & 0 & 1 & 6 & 5 & 2 \\ 4 & 3 & 5 & 1 & 0 & 2 & 7 & 6 \\ 5 & 7 & 4 & 6 & 2 & 0 & 3 & 1 \\ 6 & 2 & 1 & 5 & 7 & 3 & 0 & 4 \\ 7 & 5 & 3 & 2 & 6 & 1 & 4 & 0 \end{pmatrix}$$

Figure 3:

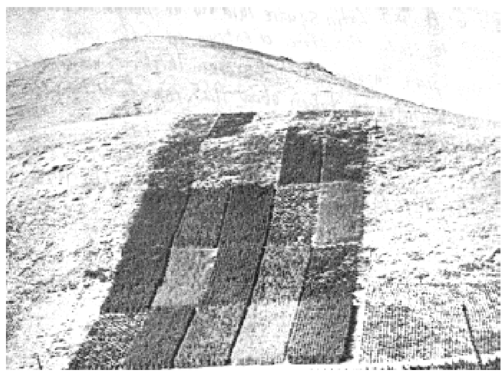


Figure 4:

Application Mutually Orthogonal Latin Squares

Latin squares: A Latin square is a $d \times d$ array where a given entry does not repeat along a row or a column:Experimental realization: fertilizers application

Mutually orthogonal squares:

- Each is a Latin square in its own right
- When superposed, no (ordered) pair of numbers repeats itself
- Any solution breaks up into two (or more) squares

Phase-space solution for primer dimensions:

$$L_{kl}^{(m)} = k + ml \pmod p$$

To each ray (except $\alpha = 0$ and $\beta = 0$) corresponds LS

LS corresponding to rays with different slope m are orthogonal: there are $p - 1$ MOLS

The resulting arrays are MOLS:

Aα	Bδ	Cβ	Dε	Eγ
Bβ	Cε	Dγ	Eα	Aδ
Cγ	Dα	Eδ	Aβ	Bε
Dδ	Eβ	Aε	Bγ	Cα
Eε	Aγ	Bα	Cδ	Dβ

Figure 5:

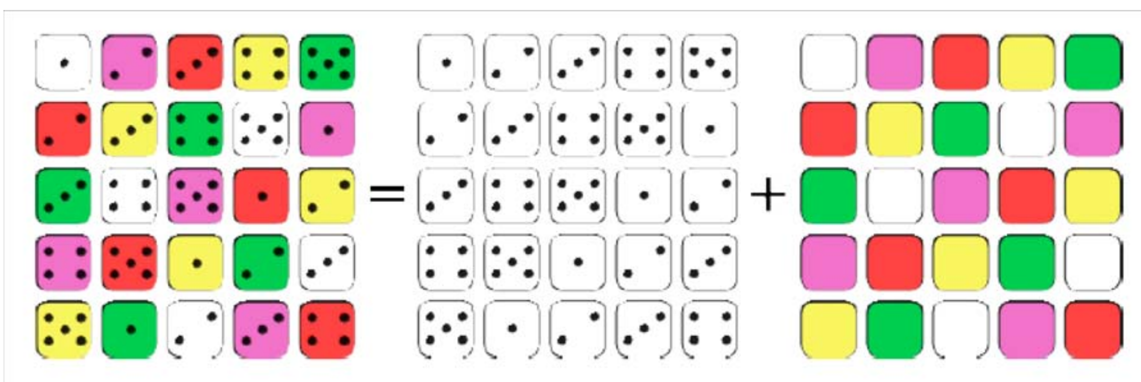
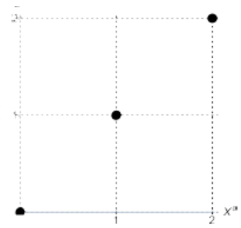


Figure 6:

$\{ZX, (ZX)^2\} \quad \rightarrow \beta = \alpha :$

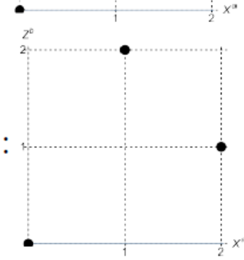


$\mapsto$

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

,

$\{ZX^2, (ZX^2)^2\} \quad \rightarrow \beta = 2\alpha :$



$\mapsto$

	0	1	2
0	0	1	2
2	2	0	1
1	1	2	0

,

Figure 7:

00	11	22
12	20	01
21	02	10

Figure 8: