Introduction to discrete calculus

Algebraic operations mod p, p-a prime number:

integers $\{0, 1, ..., p-1\}$ form a commutative group with respect to summation and $\{1, ..., p-1\}$ form a group with respect to multiplication.

a) $a \cdot b = c \in Z_p$

b) exists a e neutral element $a \cdot e = e \cdot a = a$,

c) For any a exists inverse elements $a^{-1} : a \cdot a^{-1} = a^{-1} \cdot a = e$,

where $\cdot$ means summation or multiplication.

Such structure is called an *algebraic field* Z_p .

Let us consider p - dim Hilbert space $\mathcal{H}$ and $\{|n\rangle, n = 0, ..., p-1\}$ as an orthogonal basis.

Basic operators X and Z

$$Z|n\rangle = \omega(n)|n\rangle \ \rightarrow \ Z = \sum_{n=0}^{p-1} \omega(n)|n\rangle\langle n|,$$

$$X|n\rangle = |n+1\rangle \ \rightarrow \ X = \sum_{n=0}^{p-1} |n+1\rangle\langle n|, .$$

where

$$\omega = e^{\frac{2\pi i}{p}}, \quad \omega(n) = \omega^n, \quad \sum_{n=0}^{p-1} \omega(nk) = p\delta_{k,0}.$$

X, Z are cyclic operators:

$$Z^p = X^p = I$$

Commutation relations

$$ZX|n\rangle = \omega(n+1)|n+1\rangle, \quad XZ|n\rangle = \omega(n)|n+1\rangle,$$

so that

$$ZX = \omega XZ,$$

$\{Z, X\}$ form the generalized Pauli group.

Example: Z_3

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \omega = e^{2\pi i/3}$$

Dual basis $\{|\tilde{n}\rangle\}$ where Z acts displacements

$$Z|\tilde{n}\rangle = |\widetilde{n+1}\rangle, \quad X|\tilde{n}\rangle = \omega^*(n)|\tilde{n}\rangle, \tag{1}$$

X is diagonal is diagonal in the dual basis $|\tilde{n}\rangle$

The expansion

$$|\tilde{n}\rangle = c\sum_{k=0}^{p-1}\omega(nk)|k\rangle \rightarrow \langle\tilde{n}|\tilde{n}\rangle = 1 \rightarrow c = \frac{1}{\sqrt{p}}.$$

The operator which maps $|n\rangle$ into $|\tilde{n}\rangle$,

$$|\tilde{n}\rangle = F|n\rangle, \tag{2}$$

is the finite Fourier transform

$$F = \frac{1}{\sqrt{p}}\sum_{k,n=0}^{p-1}\omega(nk)|n\rangle\langle k|, \quad FF^{\dagger} = F^{\dagger}F = I.$$

$p = 2 : k = -k$

$$F^2 = \frac{1}{2}\sum_{k,n,k'=0}^{2}\omega(k(n+k'))|n\rangle\langle k'| = I$$

$p \neq 2$

$$F^2 = \frac{1}{p}\sum_{k,n,k'=0}^{p-1}\omega(k(n+k'))|n\rangle\langle k'| = \sum_{k}^{p-1}|-k\rangle\langle k| = P,$$

P is the parity operator,

$$F^4 = P^2 = I, \tag{3}$$

The relation between X and Z via the finite Fourier transform,

$$\begin{aligned} X &= F\sum_{n=0}^{p-1}\omega(-n)|n\rangle\langle n|F^{\dagger} \\ &= FP\sum_{n=0}^{p-1}\omega(n)|n\rangle\langle n|PF^{\dagger}, \end{aligned}$$

thus

$$X = F^{\dagger}ZF. \tag{4}$$

Orthogonality relations:

$$Tr(X^n X^{\dagger m}) = p\delta_{mn}, \quad Tr(Z^n Z^{\dagger m}) = p\delta_{mn}, \quad Tr(Z^n X^m) = p\delta_{m0}\delta_{n0}.$$

The displacement operators

$$\begin{aligned} D(\alpha,\beta) &= e^{i\phi(\alpha,\beta)}Z^{\alpha}X^{\beta}, \\ D(\alpha,\beta)\,D^{\dagger}(\alpha,\beta) &= I \end{aligned}$$

form an operational basis in $\mathcal{H}$:

$$\hat{f} = \frac{1}{p} \sum_{\alpha,\beta=0}^{p-1} f_{\alpha\beta} D(\alpha,\beta). \tag{5}$$

Orthogonality relation

$$\mathrm{Tr}\left(D(\alpha,\beta)D^{\dagger}(\alpha',\beta')\right) = p\delta_{\alpha,\alpha'}\delta_{\beta,\beta'},$$

so that

$$f_{\alpha\beta} = \mathrm{Tr}\left(\hat{f}D^{\dagger}(\alpha,\beta)\right).$$

Example:

$$\begin{aligned} |k\rangle\langle k| \leftrightarrow f_{\alpha\beta}(k) &= \mathrm{Tr}\left[|k\rangle\langle k|D^{\dagger}(\alpha,\beta)\right] = \\ &= e^{-i\phi(\alpha,\beta)}\omega(-k\alpha)\delta_{\beta,0}, \end{aligned}$$

Drawback: $|k\rangle\langle k|$ is Hermitian, but $f_{\alpha\beta}(k)$ is a complex function

This means that we can map $\hat{f} \in \mathrm{Op}(\mathcal{H})$ into $f_{\alpha,\beta}$ which is a function of discrete variables defined on a discrete 2-dim space M. The coordinates (α,β) in M are given by the powers of Z and X, respectively. Due to the periodicity of Z and X the space M is diffeomorphic to a bidimensional discrete torus. The action of the operator $D(\alpha',\beta')$ on an arbitrary point (α,β) on the manifold is just the displacement $(\alpha+\alpha',\beta+\beta')$, for this reason $D(\alpha',\beta')$ is called a *displacement operator*.

1 Discrete phase space geometry

The discrete phase-space: collection of points of two-dim grid $(\alpha,\beta) \in Z_p \times Z_p$

Lines:

$$a\alpha + b\beta = c,\ a,b,c \in Z_p$$

Equation

$$a\alpha = c$$

has a unique solution on Z_p

$$\alpha = a^{-1}c.$$

Parallel lines:

$$\begin{aligned} a\alpha + b\beta &= c \\ a'\alpha + b'\beta &= c', \end{aligned}$$

with

$$\frac{b}{a} = \frac{b'}{a'} \rightarrow ba' = ab'.$$

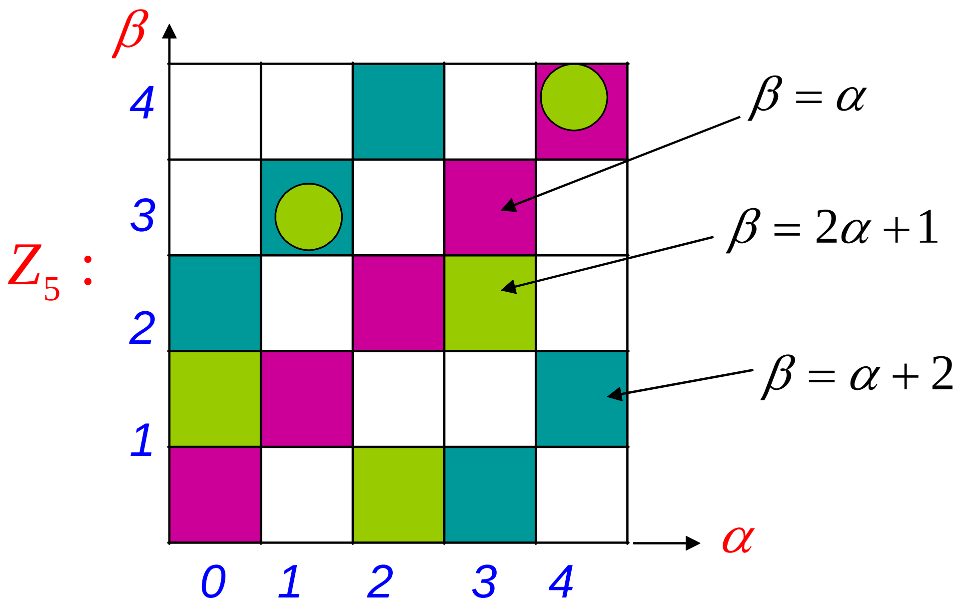

Figure 1:

have no common points.

If the lines are not parallel they cross at a single point with the coordinates

$$\alpha = \frac{c' - b'b^{-1}c}{a' - ab'b^{-1}}, \quad \beta = \frac{c' - a'a^{-1}c}{b' - a'a^{-1}b}.$$

Ray is a line which passes through the origin

$$\beta = m\alpha, \quad \text{or} \quad \alpha = 0.$$

There are $p-1$ parallel lines to each of $p+1$ rays: the total number of lines is $p(p+1)$.

Points of each ray form an Abelian group:

$$(\alpha, m\alpha) + (\alpha', m\alpha') = (\alpha + \alpha', m(\alpha + \alpha'))$$

The collection of p parallel lines is called *striation*.

Displacement in the discrete phase space

General association:

$$(\alpha, \beta) \Leftrightarrow D(\alpha, \beta)$$

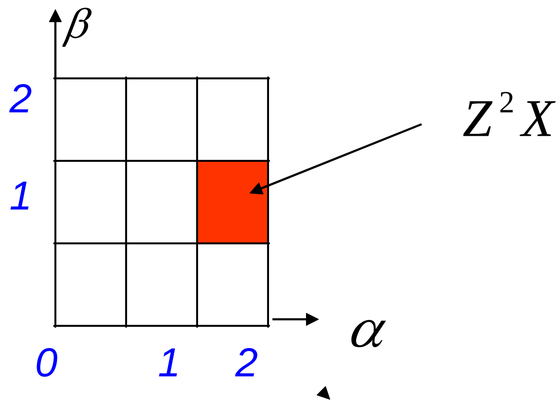

Figure 2:

Example: $\mathcal{Z}_3$
The displacement operators corresponding to the same ray commute

$$\beta = m\alpha \Leftrightarrow [D(\alpha, m\alpha), D(\alpha', m\alpha')] = 0,$$

$$\alpha = 0 \Leftrightarrow [D(0,\beta), D(0,\beta') = 0.$$

Example: all the possible rays for $\mathcal{Z}_3$

$$\begin{array}{rcl} \beta = 0 & \rightarrow & Z, Z^2 \\ \beta = \alpha & \rightarrow & ZX, Z^2X^2 \\ \beta = 2\alpha & \rightarrow & ZX^2, Z^2X^4 \\ \alpha = 0 & \rightarrow & X, X^2. \end{array}$$

The set $\{Z^\alpha X^{m\alpha}\}$ with fixed m has p different eigenvectors $|\psi_m^n\rangle$, $n = 0, ..., p-1$
Let us associate the ray $\beta = m\alpha$ with an eigenstate $|\psi_m^0\rangle$ of $D(\alpha, m\alpha)$

$$D(\alpha, m\alpha)|\psi_m^0\rangle = e^{i\xi_m^0}|\psi_m^0\rangle$$

all the other eigenstates are related to $|\psi_m^0\rangle$

$$|\psi_m^n\rangle = X^n|\psi_m^0\rangle,$$

$$D(\alpha, m\alpha)|\psi_m^n\rangle = e^{i\phi(\alpha,\beta)}Z^\alpha X^{m\alpha}X^n|\psi_m^0\rangle = \omega(n\alpha)e^{i\xi_m^0}|\psi_m^n\rangle,$$

$|\psi_m^n\rangle$ is associated with the line $\beta = m\alpha + n$,

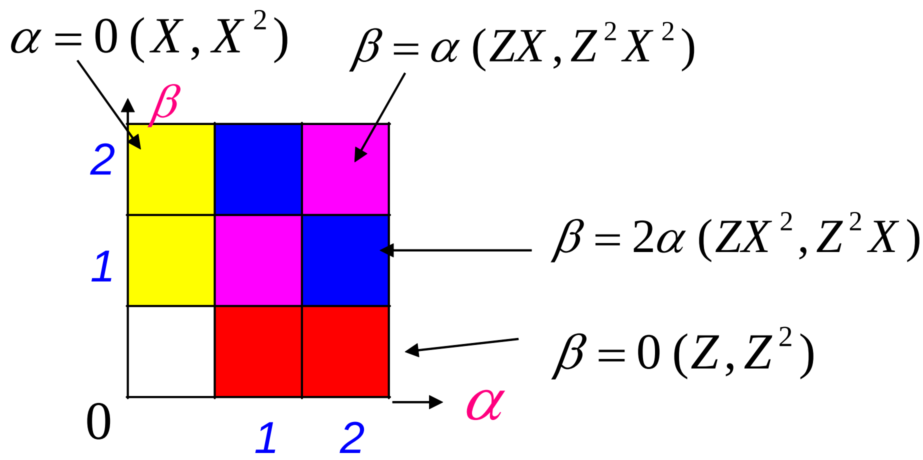

Figure 3:

Orthogonality of two states with indices belonging to the same foliation

$$\langle \psi_m^n | \psi_m^{n'} \rangle = \delta_{n,n'}.$$

Rotations in the phase space

Rotation operator V

$$V Z^\alpha V^\dagger = \varphi Z^\alpha X^\alpha, \tag{6}$$

where φ is a phase factor and

$$[V, X] = 0. \tag{7}$$

Since V is diagonal in the basis $\{|\tilde{n}\rangle\}$

$$V = \sum_{n=0}^{p-1} c_n |\widetilde{n}\rangle\langle\widetilde{n}|, \quad c_0 = 1. \tag{8}$$

The left-hand side of 6)

$$\begin{aligned} V Z^\alpha V^\dagger &= \sum_{n,n'=0}^{p-1} c_{n'} c_n^* |\widetilde{n}'\rangle\langle\widetilde{n}'|Z^\alpha|\widetilde{n}\rangle\langle\widetilde{n}| \\ &= \sum_{n=0}^{p-1} c_{n+\alpha} c_n^* |\widetilde{n+\alpha}\rangle\langle\widetilde{n}|, \end{aligned}$$

The right -hand side of (6)

$$Z^\alpha X^\alpha = Z^\alpha X^\alpha \sum_{n=0}^{p-1} |\widetilde{n}\rangle\langle\widetilde{n}| = \sum_{n=0}^{p-1} \omega(-n\alpha)|\widetilde{n+\alpha}\rangle\langle\widetilde{n}|,$$

the equation for c_n

$$\begin{aligned} c_{n+\alpha}c_n^* &= \varphi\omega(-n\alpha), \rightarrow |c_n|^2 = 1, & (9) \\ & & (10) \\ c_\alpha &= \varphi = \omega(-2^{-1}\alpha^2). & (11) \end{aligned}$$

A particular solution for $p \neq 2$

$$c_n = \omega(-2^{-1}n^2),$$

thus

$$\begin{aligned} V &= \sum_{n=0}^{p-1} \omega(-2^{-1}n^2)|\widetilde{n}\rangle\langle\widetilde{n}|, \\ VZ^\alpha V^\dagger &= \omega(-2^{-1}\alpha^2)Z^\alpha X^\alpha. \end{aligned}$$

According to that we get

$$V^m Z^\alpha (V^\dagger)^m = \omega(-2^{-1}m\alpha^2)Z^\alpha X^{m\alpha}. \tag{12}$$

Example: $\mathcal{Z}_3$

$$\omega = \exp(2i\pi/3)$$

$$V = |\widetilde{0}\rangle\langle\widetilde{0}| + \exp(2i\pi/3)|\widetilde{1}\rangle\langle\widetilde{1}| + \exp(-2i\pi/3)|\widetilde{2}\rangle\langle\widetilde{2}|$$

From the geometric point of view powers of V^m produces rotations from one ray to another one:

$$\underbrace{\lambda_0 \xrightarrow{V} \lambda_1 \xrightarrow{V} \lambda_2}_{V^2},$$

$$V^m : (\alpha, 0) \rightarrow (\alpha, m\alpha) : \lambda_0 \rightarrow \lambda_m,$$

i.e. from the horizontal ray $\beta = 0$ one can obtain all the rays except $\alpha = 0$

More generally, for odd p:

$$D(\alpha, (m+1)\alpha)[V|\psi_m^n\rangle] = \omega(n\alpha)e^{i\xi_m^0}[V|\psi_m^n\rangle],$$

the states $V|\psi_m^n\rangle$ are associated with the striation $\beta = (m+1)\,\alpha$

For $p = 2$

$$c_{n+1}c_n^* = \varphi\omega(-n), \quad c_0 = 1, \quad c_1 = \varphi = \pm i,$$

and (6) reads

$$VZV^\dagger = \left\| \begin{matrix} 1 & 0 \\ 0 & i \end{matrix} \right\| Z \left\| \begin{matrix} 1 & 0 \\ 0 & -i \end{matrix} \right\| = iZX.$$

V is a cyclic operator

$$V^p = I.$$

One cannot reach $(0,\beta)$ from $(\alpha,0)$. The transformation $(0,\beta) \to (\alpha,0)$ is performed by the Fourier transform

$$X^\alpha = F^\dagger Z^\alpha F$$

The operator

$$U = FVF^\dagger = \sum_{k=0}^{p-1} c_{-n} |n\rangle\langle n|, \tag{13}$$

transforms

$$UZ^\alpha X^{m\alpha} U^\dagger \sim Z^{(1-m)\alpha} X^{m\alpha},$$

and allows to obtain from the ray $\alpha = 0$ any other ray except $\beta = 0$

$$\beta = m\alpha \xrightarrow{U} (1-m)\beta = m\alpha,$$

Discrete squeezing operator

$$S_\lambda = \sum_\kappa |\kappa\rangle\langle\lambda\kappa| \,. \tag{14}$$

The following relations hold

$$S_\lambda^\dagger Z_\alpha S_\lambda = Z^{\alpha\lambda^{-1}}\,, \qquad\qquad S_\lambda^\dagger X_\alpha S_\lambda = X^{\alpha\lambda}\,, \tag{15}$$

The operators V, U, S are elements of the Clifford group $\mathcal{C}$, which is a stabilizer of the Heisenberg-Weyl group, i.e, HW group is an invariant subgroup of the HW group

$$\mathcal{C}D(\alpha,\beta)\mathcal{C}^{-1} \sim D(\gamma,\delta)$$

2 Discrete Wigner function

Construct the kernel as a double Fourier transform of the displacement operator:

$$\hat{w}(\alpha,\beta) = \frac{1}{p} \sum_{\gamma,\delta=0}^{p-1} \omega(\alpha\delta - \beta\gamma) D(\gamma,\delta), \tag{16}$$

the orthogonality relation

$$\mathrm{Tr}\left(\hat{w}(\alpha,\beta)\hat{w}^\dagger(\alpha',\beta')\right) = p\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}. \tag{17}$$

The kernels (16) form an operational basis

$$f=\sum_{\alpha,\beta=0}^{p-1} W_f(\alpha,\beta)\hat{w}(\alpha,\beta)\leftrightarrow W_f(\alpha,\beta)=\frac{1}{p}\mathrm{Tr}\left(\hat{f}\hat{w}(\alpha,\beta)\right). \tag{18}$$

Let us impose the condition

$$f^\dagger \leftrightarrow W_f^*(\alpha,\beta)$$

so that

$$f^\dagger=\sum_{\alpha,\beta=0}^{p-1} W_f^*(\alpha,\beta)\hat{w}^\dagger(\alpha,\beta)=\sum_{\alpha,\beta=0}^{p-1} W_f^*(\alpha,\beta)\hat{w}(\alpha,\beta)$$

therefore $\hat{w}$ has to be a Hermitian operator.

$$\hat{w}=\hat{w}^\dagger$$

This also gives a phase condition

$$\hat{w}^\dagger=\frac{1}{p}\sum_{\gamma,\delta=0}^{p-1}\omega(\alpha\delta-\beta\gamma)e^{-i\phi(-\gamma,-\delta)}\omega(-\gamma\delta)Z^\gamma X^\delta,$$

so

$$\omega(-\gamma\delta)=e^{i\phi(\gamma,\delta)+i\phi(-\gamma,-\delta)}. \tag{19}$$

A particular solution of (19) is for odd primes

$$\phi(\gamma,\delta)=\omega(-2^{-1}\gamma\delta),$$

for $p=2$,

$$\phi(\gamma,\delta)=(\pm i)^{-\gamma\delta}\,.$$

$\hat{w}(\alpha,\beta)$ is covariant

$$D(\mu,\nu)\hat{w}(\alpha,\beta)D^\dagger(\mu,\nu)=\hat{w}(\alpha+\mu,\beta+\nu).$$

The symbol of a transformed operator

$$\hat{f}=D(\mu,\nu)fD^\dagger(\mu,\nu),$$

has the form

$$\begin{aligned} W_{\hat{f}}(\alpha,\beta) &= \frac{1}{p}\mathrm{Tr}(D(\mu,\nu)fD^\dagger(\mu,\nu)\hat{w}(\alpha,\beta)) \\ &= W_f(\alpha-\mu,\beta-\nu). \end{aligned}$$

An explicitly covariant representation of (16) is

$$\hat{w}(\alpha,\beta)=\frac{1}{p}D(\alpha,\beta)\left[\sum_{\gamma,\delta=0}^{p-1}D(\gamma,\delta)\right]D^\dagger(\alpha,\beta).$$

For odd p:

$$\frac{1}{p}\sum_{\gamma,\delta=0}^{p-1} D(\gamma,\delta) = \frac{1}{p}\sum_{k=0}^{p-1}\sum_{\gamma,\delta=0}^{p-1} \omega((k+2^{-1}\delta)\gamma)|k+\delta\rangle\langle k| = P,$$

The normalization condition for (16)

$$\mathrm{Tr}\hat{w}(\alpha,\beta) = \frac{1}{p}\sum_{\gamma,\delta=0}^{p-1} \omega(\alpha\delta-\gamma\beta)\phi(\gamma,\delta)\mathrm{Tr}(Z^\gamma X^\delta) = 1$$

as $\phi(0,0)=1$.

Thus

$$\mathrm{Tr} f = \sum_{\alpha,\beta=0}^{p-1} W_f(\alpha,\beta).$$

Note that if

$$f = \sum_{\alpha,\beta=0}^{p-1} f_{\alpha\beta} D(\alpha,\beta),$$

then the symbol of the operator $\hat{f}$ can be obtained as

$$\begin{aligned} W_f(\alpha,\beta) &= \frac{1}{p}\sum_{\mu,\nu=0}^{p-1} f_{\mu\nu}\mathrm{Tr}\left(D(\mu,\nu)\hat{w}(\alpha,\beta)\right) \\ &= \sum_{\mu,\nu=0} f_{\mu\nu}\omega(-\alpha\nu+\beta\mu). \end{aligned} \tag{20}$$

The trace condition leads to

$$\mathrm{Tr}(fg) = p\sum_{\alpha,\beta=0}^{p-1} W_f(\alpha,\beta)W_g(\alpha,\beta).$$

Examples:

$$\begin{aligned} W_{|n\rangle\langle n|} &= \frac{1}{p}\sum_{\gamma,\delta=0}^{p-1} \omega(\alpha\delta-\beta\gamma)\mathrm{Tr}\left(|k\rangle\langle k|D(\gamma,\delta)\right) \\ &= \frac{1}{p}\sum_{\gamma,\delta=0}^{p-1} \omega(\alpha\delta-\beta\gamma)\phi(\gamma,0)\omega(n\gamma)\delta_{\delta,0} = \delta_{\beta,n}; \end{aligned}$$

$$\begin{aligned} W_Z &= \frac{1}{p}\sum_{\gamma,\delta=0}^{p-1} \omega(\alpha\delta-\beta\gamma)\mathrm{Tr}(ZD(\gamma,\delta)) \\ &= \sum_{\gamma,\delta=0}^{p-1} \omega(\alpha\delta-\beta\gamma)\delta_{\delta,0}\delta_{\gamma,-1} = \omega(\beta), \end{aligned}$$

3 General maps

Define "discrete coherent states"

$$|\alpha,\beta\rangle = D(\alpha,\beta)|\psi_0\rangle$$

consider the projector

$$\begin{aligned}\hat{w}^{(-1)}(\alpha,\beta) &= |\alpha,\beta\rangle\langle\alpha,\beta| \\ \sum_{\alpha,\beta}|\alpha,\beta\rangle\langle\alpha,\beta| &= p\hat{I}\end{aligned}$$

$$\begin{aligned}D(\alpha,\beta)|\psi_0\rangle\langle\psi_0|D^{\dagger}(\alpha,\beta) &= \frac{1}{p}\sum_{m,n} tr\left[D(\alpha,\beta)|\psi_0\rangle\langle\psi_0|D^{\dagger}(\alpha,\beta)D(m,n)\right]D(m,n) \\ &= \frac{1}{p}\sum_{m,n} tr\left[\langle\psi_0|D^{\dagger}(\alpha,\beta)D(m,n)D(\alpha,\beta)|\psi_0\rangle\right]D(m,n) \\ &= \frac{1}{p}\sum_{m,n} e^{i\phi(m,n)} tr\left[\langle\psi_0|X_\beta Z_\alpha Z_m X_n Z_\alpha X_\beta|\psi_0\rangle\right]D(m,n) \\ &= \frac{1}{p}\sum_{m,n}\omega(\alpha n+\beta m)\, tr\left[\langle\psi_0|D(m,n)|\psi_0\rangle\right]D(m,n) = \hat{w}^{(-1)}(\alpha,\beta)\end{aligned}$$

map generated by $\hat{w}^{(-1)}(\alpha,\beta)$

$$Q_f(\alpha,\beta) = Tr(\hat{f}\hat{w}^{(-1)}(\alpha,\beta)) = \langle\alpha,\beta|\hat{f}|\alpha,\beta\rangle,$$

Inverse expansion

$$\begin{aligned}\hat{f} &= \sum_{\alpha,\beta} P_f(\alpha,\beta)|\alpha,\beta\rangle\langle\alpha,\beta|, \\ P_f(\alpha,\beta) &= tr\left[\hat{f}\hat{w}^{(s=1)}(\alpha,\beta)\right]\end{aligned}$$

so that

$$Tr(\,\hat{f}\hat{g}) = \sum_{\alpha,\beta} P_f(\alpha,\beta)Q_g(\alpha,\beta)$$

General map

$$\hat{w}^{(s)}(\alpha,\beta) = \frac{1}{p}\sum_{\gamma,\delta}(-1)^{\alpha\delta+\beta\gamma}\left[\langle\xi|D(\gamma,\delta)|\xi\rangle\right]^{-s}D(\gamma,\delta), \tag{21}$$

Traciality relation

$$\mathrm{Tr}\left(\hat{w}^{(s)}(\alpha,\beta)\hat{w}^{\dagger(-s)}(\alpha',\beta')\right) = p\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$$

$$\hat{D}(m,n)\,\hat{w}^{(s)}(\alpha,\beta)\,\hat{D}^{\dagger}(m,n) = \hat{w}^{(s)}(\alpha+m,\beta+n)\,. \tag{22}$$

Covariant form

$$\hat{w}^{(s)}(\alpha,\beta) = \hat{D}(\alpha,\beta)\,\hat{w}^{(s)}(0,0)\,\hat{D}^{\dagger}(\alpha,\beta)\,, \tag{23}$$

where

$$\hat{w}^{(s)}(0,0) = \frac{1}{d}\sum_{\alpha,\beta}\hat{D}(\alpha,\beta)\,\langle\psi_0|\hat{D}(\alpha,\beta)|\psi_0\rangle^{-s}\,, \tag{24}$$

Fiducial state: symmetric states

$$\left|\langle\psi_0|Z^{\alpha}X^{\beta}\,|\psi_0\rangle\right|^2 = \frac{1+p\delta_{\alpha,0}\delta_{\beta,0}}{1+p}, \tag{25}$$

then overlap of two DCS

$$\left|\ \alpha,\beta|\alpha',\beta'\rangle\right|^2 = \frac{1+p\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}}{1+p}. \tag{26}$$

Transformation under discrete symplectic operations

$$W_{V_m\rho V_m^{\dagger}}\,(a,b) = W_{\rho}\,(a,b-ma) \tag{27}$$

$$W_{U_m\rho U_m^{\dagger}}\,(a,b) = W_{\rho}\,(a-mb,b) \tag{28}$$

$$W_{S_q\rho S_q^{\dagger}}\,(a,b) = W_{\rho}\left(qa,q^{-1}b\right) \tag{29}$$