Classical Hamiltonian mechanics

CHM: manifold $\mathcal{M}$ + *Poisson structure*
x^i - local coordinates in $\mathcal{M}$
$\omega^{jk} = -\omega^{kj}$ define *Poisson brackets*:
if f and g are functions on $\mathcal{M}$

$$\{f, g\} = \omega^{jk} \partial_j f \partial_k g$$

if $\omega^{jk}$is non-degenerated $\Rightarrow$ exist $\omega_{jk} : \omega^{jk}\omega_{jl} = \delta^k_l$
and 2-form is defined

$$\omega = \omega_{jk}(x) dx^j \wedge dx^k$$

Being $\mathcal{M}$ a differentiable manifold of dimension $2n$.
Symplectic form on $\mathcal{M}$ is a closed non-degenerated 2-form:

$$\det ||\omega^{jk}|| \neq 0, \quad d\omega = 0$$

($\mathcal{M}$,ω) is called *symplectic manifold*

Darboux Theorem: in a symplectic manifold one can introduce local coordinates (p, q) such that:

$$\omega = dp_j \wedge dq^k,$$

$$\{p_j, q^k\} = \delta^k_l, \quad \{p_j, p_k\} = \{q^j, q^k\} = 0$$

Classical observable: a real smooth function on $\mathcal{M}$

Symmetry group

Being G is a Lie group and $\mathcal{M}$ is a symplectic manifold.

Action G on $\mathcal{M}$ is a smooth application ψ:
$G \times \mathcal{M} \to \mathcal{M}$ such that for any $x \in \mathcal{M}$ and every $g \in G$

1. $\psi(g, x) \in \mathcal{M}$
2. $\psi(e, x) = x$
3. $\psi(g_1, \psi(g_2 x)) = \psi(g_1 g_2, x)$

G acts transitively on $\mathcal{M}$

$$x, x' \in \mathcal{M} \Rightarrow x' = g \cdot x$$

Quantization problem

Dirac quantization rule

Find a linear correspondence:

$$F^{\infty}(\mathcal{M}) \rightarrow \mathrm{Op}(\mathcal{H})$$

$F^{\infty}(\mathcal{M})$ - set of classical observables on $\mathcal{M}$
H - a Hilbert space
$\mathrm{Op}(H)$ - set of Hermitian operators acting in H

$$f \rightarrow \hat{f}, \quad g \rightarrow \hat{g}, \quad \hat{f}, \hat{g} \in \mathrm{Op}(\mathcal{H})$$

that satisfy the correspondence rule (commutation relation)

$$\{f, g\} \rightarrow -\frac{i}{\hbar}[\hat{f}, \hat{g}]$$

Ejample: $\mathcal{M} = \mathcal{R}^2$
Linear functions:

$$q \rightarrow \hat{q}, \qquad p \rightarrow \hat{p}, \qquad [\hat{q}, \hat{p}] = i\hbar$$

Quadratic functions:

$$q^2 \rightarrow \hat{q}^2, \qquad p^2 \rightarrow \hat{p}^2$$

$$qp = \frac{1}{4}\left((q+p)^2 - (q-p)^2\right) \rightarrow \frac{1}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)$$

cubic functions

$$qp^2 \rightarrow ??$$

Symmetry quantization
Being G a symmetry group of $\mathcal{M}$, $\mathcal{H}$ is a Hilbert space where G acts irreducible by unitary are operators T_g.

For every f on $\mathcal{M}$ find the correspondence

$$f(x) \to \hat{f}, \quad \hat{f} \in \mathrm{Op}(\mathcal{H})$$

that preserve the commutation relations and is covariant under action of G:

$$f(g^{-1} \cdot x) \to T_g \, \hat{f} \, T_g^{\dagger},$$

Weyl quantization
Consider $\mathcal{M} = \mathcal{R}^2$. Quantization $\mathcal{R}^2 \to \mathcal{H} = L_2(\mathcal{R}^1)$

Linear functions

$$q \to \hat{q}, \qquad p \to \hat{p}, \qquad [\hat{q}, \hat{p}] = i\hbar$$

$$\hat{q}\psi(\xi) = \xi\psi(\xi), \quad \hat{p}\psi(\xi) = -i\hbar\partial_\xi\psi(\xi), \quad \psi(\xi) \in L_2(\mathcal{R}^1)$$

Arbitrary functions

$$f(p,q) = \frac{1}{(2\pi)^2} \int dxdy e^{-ixq-iyp} \int dq'dp' e^{ixq'+iyp'} f(p',q')$$

Quantization

$$e^{-ixq-iyp} \to \hat{E}(x,y) = \begin{cases} e^{-ix\hat{q}-iy\hat{p}} \\ e^{-ix\hat{q}} e^{-iy\hat{p}} \\ e^{-iy\hat{p}} e^{-ix\hat{q}} \\ \ldots \end{cases}$$

so that

$$f(p,q) \to \hat{f} = \frac{1}{(2\pi)^2} \int dxdy \hat{E}(x,y) \int dqdp e^{ixq+iyp} f(p,q)$$

Weyl form

$$\hat{f} = \frac{1}{2\pi} \int dq dp \hat{w}(p,q) f(p,q)$$

where

$$\hat{w}(p,q) = \frac{1}{2\pi} \int dx dy \hat{E}(x,y) e^{ixq+iyp}$$

$\hat{w}(\xi)$, $\xi = (p,q)$ - a quantizer kernel $\in \mathrm{Op}(\mathcal{H})$

$\xi \in \mathcal{R}^2$ symmetry group is $H(1) : g = (\mu, \nu; z) \in H(1)$

$$g = (x, y; z) = \exp\left(izI - \frac{i}{\hbar} x\hat{q} - \frac{i}{\hbar} y\hat{p} \right)$$

Action of $H(1)$ on $\mathcal{R}^2 = \{(p,q)\}$

$$g^{-1}(p,q) = (p - \mu, q + \nu,)$$

Symplectic structure on $\mathcal{R}^2$

$$\begin{aligned} \{f, g\} &= \partial_p f \, \partial_q g - \partial_q f \, \partial_p g \\ \{p, q\} &= 1, \quad 2 - form \ \omega = dp \wedge dq \end{aligned}$$

Properties of $\hat{w}(\xi)$

1. Covariance: $T_g \hat{w}(p,q) T_g^\dagger = \hat{w}(g \cdot \xi)$, $g^{-1} \cdot \xi = (p - \mu, q + \nu)$
2. Hermiticity: $\hat{w}(p,q) = \hat{w}^\dagger(p,q)$
3. Normalization: $Tr \hat{w}(p,q) = 1$
4. Traciality: $Tr\left(\hat{w}(p,q)\hat{w}(p',q')\right) = 2\pi \delta\left(p - p'\right) \delta\left(q - q'\right)$

Inversion problem
The traciality

$$Tr\left(\hat{w}(p,q)\hat{w}(p',q')\right) = 2\pi\delta\left(p-p'\right)\delta\left(q-q'\right)$$

allows to invert the quantization

$$\hat{f} = \frac{1}{2\pi}\int dqdp\hat{w}(p,q)f(p,q)$$

giving

$$f(p,q) = Tr\left(\hat{w}(p,q)\hat{f}\right) = W_f(p,q)$$

$W_f(p,q)$ - is a symbol of $\hat{f}$

Isomorphism generated by $\hat{w}(p,q)$

$$\hat{f} \overset{\hat{w}(p,q)}{\leftrightarrow} W_f(p,q)$$

Properties of $W_f(\xi = (p,q))$

1. $W_{f_g}(\xi) = W_f(g^{-1}\cdot\xi)$, $\hat{f}_g = T_g\hat{f}T_g^\dagger$
2. $W_I(\xi) = I$
3. $W_{f^\dagger}(\xi) = W_f^*(\xi)$
4. $Tr\left(\hat{f}\hat{g}\right) = 1/2\pi\int d^2\xi W_f(\xi)W_g(\xi), \quad d^2\xi = dqdp$

Particular case: $\hat{f} = \rho$ - density matrix
$W_\rho(\xi)$ - *quasidistribution function*
$W_\rho(\xi)$ contains the same information as ρ

Moyal-Weyl quantization program

Given $(\mathcal{M}, G, H)$
$\mathcal{M}$ -a symplectic manifold (classical phase-space)
G - an invariance group of $\mathcal{M}$
H - a Hilbert space
establish a linear map

$$F^{\infty}(\mathcal{M}) \rightarrow \mathrm{Op}(H)$$

generated by a kernel $\hat{w}(\xi) \in \mathrm{Op}(H)$, $\xi \in \mathcal{M}$ such that

1. $T_g \hat{w}(\xi) T_g^{\dagger} = \hat{w}(g \cdot \xi)$, $g \in G$
2. $\hat{w}(\xi) = \hat{w}(\xi)^{\dagger}$
3. $Tr \hat{w}(\xi) = 1$
4. $Tr\left(\hat{w}(\xi)\hat{w}(\xi')\right) = \Delta(\xi, \xi')$

where $\Delta(\xi, \xi')$ - is a reproductive kernel:

$$\int_{\mathcal{M}} d\mu(\xi) \Delta(\xi, \xi') g(\xi) = g(\xi')$$

being $d\mu(\xi)$ - invariant measure on $\mathcal{M}$
If such $\hat{w}(\xi)$ exists, then
the quantization procedure

$$f(\xi) \rightarrow \hat{f} = \int_{\mathcal{M}} d\mu(\xi) \hat{w}(\xi) f(\xi)$$

the de-quantization procedure

$$\hat{f} \rightarrow W_f(\xi) = Tr\left(\hat{w}(\xi)\hat{f}\right)$$

Coherent states of a group G

Being T_g unitary irrep of G in a Hilbert space $\mathcal{H}$.
An orbit $|\psi_g\rangle$ of a state $|\psi_0\rangle \in \mathcal{H}$ is

$$|\psi_g\rangle = T_g|\psi_0\rangle, \quad \text{for all } g \in G$$

A stationary subgroup $G_0 \subset G$ of $|\psi_0\rangle$

$$T_{g_0}|\psi_0\rangle = e^{i\phi(g_0)}|\psi_0\rangle, \quad g_0 \in G_0$$

Coherent states of G: is a set of orbits

$$\begin{aligned} |\xi_g\rangle &= T_{\tilde{g}}|\psi_0\rangle, \quad \tilde{g} \in G/G_0 \\ D(\xi) &\equiv T_{\tilde{g}} \end{aligned}$$

which means identification

$$e^{i\phi(g_0)}|\psi_g\rangle \equiv |\xi_g\rangle, \quad \text{para for all } \phi(g_0)$$

The coset space G/G_0 is a homogeneous space for G, $\xi_g \in G/G_0$

Properties of CS

1. Stability: $T_g|\xi\rangle = e^{i\phi(g)}|\xi\rangle$
2. Completness: $\int_{G/G_0} d\mu(\xi)|\xi\rangle\langle\xi| = I$

where $d\mu(\xi)$ is the invariant measure on G/G_0
Choice of the fiducial state $|\psi_0\rangle$ - G_0 is the maximal subgroup of G

Ejamples

Heisenberg-Weyl group H(1)
the basis of irrep $|n\rangle, n = 0, 1, 2, ..$
fiducial state $|\psi_0\rangle = |0\rangle$, $a|0\rangle = 0$
invariant subgroup $G_0 = (0, 0; z) = \{e^{izI}\}$
the manifold $H(1)/G_0 = \mathcal{C} = (\alpha, \alpha^*)$,
invariant measure $d\mu(\xi) = d^2\alpha/\pi$
Representation operator over $\mathcal{C}$:

$$T_{\mathcal{C}} = D(\alpha) = \exp(\alpha a^\dagger - \alpha a)$$

Coherent state

$$|\xi\rangle = |\alpha\rangle = D(\alpha)|0\rangle$$

SU(2) group
the basis of irrep dim $= 2S + 1$: $|k, S\rangle, k = -S, ..., S$
fiducial state $|\psi_0\rangle = | - S, S\rangle$
invariant subgroup $G_0 = U(1) = \{\exp(i\phi S_z)\}$
manifold $SU(2)/U(1) = \mathcal{S}^2$ (θ, φ),
invariant measure $d\mu = (2S + 1)/4\pi \sin\theta d\theta d\phi$
Representation operator over $\mathcal{S}^2$:

$$D(\theta, \phi) = \exp\left[-\frac{\theta}{2}\left(S_+ e^{-i\phi} - S_- e^{i\phi}\right)\right]$$

Coherent states

$$|\theta, \phi\rangle = D(\theta, \phi)|S, -S\rangle$$

$$\text{Ordering problem in Op}(H)$$
$$\Downarrow$$
$$s\text{-ordered kernels : } \hat{w}_s(\xi)$$
$$\Downarrow$$
$$s\text{-ordered symbols: } W_f^{(s)}(\xi)$$

ordering examples

Being $|\xi\rangle$ - CS of G

1. $P_f(\xi)$- covariant symbols (Glauber) ($s = 1$)

$$\hat{f} \to P_f(\xi), \quad \hat{f} = \int d\mu(\xi) P_f(\xi) |\xi\rangle\langle\xi|$$

2. $Q_f(\xi)$ - contravariant symbols (Husimi) ($s = -1$)

$$\hat{f} \to Q_f(\xi), \quad Q_f(\xi) = \langle\xi|\hat{f}|\xi\rangle$$

so that

$$Tr(\hat{f}\hat{g}) = \int_{\mathcal{M}} d\mu(\xi) P_f(\xi) Q_g(\xi) = \int_{\mathcal{M}} d\mu(\xi) Q_f(\xi) P_g(\xi)$$

3. $W_f(\xi)$ - self-dual symbols (Wigner) ($s = 0$)

$$\hat{f} \to W_f(\xi)$$

so that

$$Tr(\hat{f}\hat{g}) = \int_{\mathcal{M}} d\mu(\xi) W_f(\xi) W_g(\xi)$$

A map

$$\hat{f} \quad \leftrightarrow \quad W_f^{(s)}(\xi)$$

is generated by s-ordered kernel $\hat{w}_s(\xi)$

$$W_f^{(s)}(\xi) \quad = \quad \mathrm{Tr}\left(\hat{f}\hat{w}_s(\xi)\right),$$

$$\hat{f} \quad = \quad \int_{\mathcal{C}} d\mu(\xi)\hat{w}_{-s}(\xi)W_f^{(s)}(\xi)$$

so that

$$\mathrm{Tr}\left(\hat{g}\hat{f}\right) \quad = \quad \int_{\mathcal{M}} d\mu(\xi)W_g^{(s)}(\xi)\,W_f^{(-s)}(\xi)$$

General form of the kernel

$$\begin{aligned} \hat{w}_s(\xi) &= D(\xi)\hat{P}_s D^\dagger(\xi) \\ T_g \hat{P}_s T_g^\dagger &= \hat{P}_s, \quad g_0 \in G_0 \end{aligned}$$

Example: $G = H(1)$, $\mathcal{M} = \mathcal{C}, \mathcal{H} = \{|n\rangle, n = 0, 1, 2...\}$

$$\begin{aligned} \hat{w}_s(\alpha) &= \frac{1}{\pi}\int d^2\lambda D(\xi) \exp\left[\lambda^*\alpha - \lambda\alpha^* + s|\lambda|^2/2\right] \\ D(\alpha) &= \exp(\alpha a^\dagger - \alpha a) \\ \hat{w}_s(\alpha) &= D(\alpha)\hat{P}_s D^\dagger(\alpha) \\ \hat{P}_s &= \frac{1}{\pi}\int d^2\lambda D(\xi) \exp\left[s|\lambda|^2/2\right] = \frac{2}{1-s}\left(\frac{s+1}{s-1}\right)^{a^\dagger a} \\ \hat{P}_0 &= (-1)^{a^\dagger a} \end{aligned}$$