

# **Theory of Superconductivity: Two tasks**

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**ЛЕКЦІЇ**  
**З ФІЗИКИ НАДПРОВІДНОСТІ**  
(електронна версія)

**Київ 2011**

**[http://bitp.kiev.ua/files/doc/lectures/lecture\\_01.pdf](http://bitp.kiev.ua/files/doc/lectures/lecture_01.pdf)**

P. Müller A.V. Ustinov (Eds.)

V. V. Schmidt

# The Physics of Superconductors

Introduction to Fundamentals  
and Applications

with 114 Figures  
and 51 Problems with Solutions

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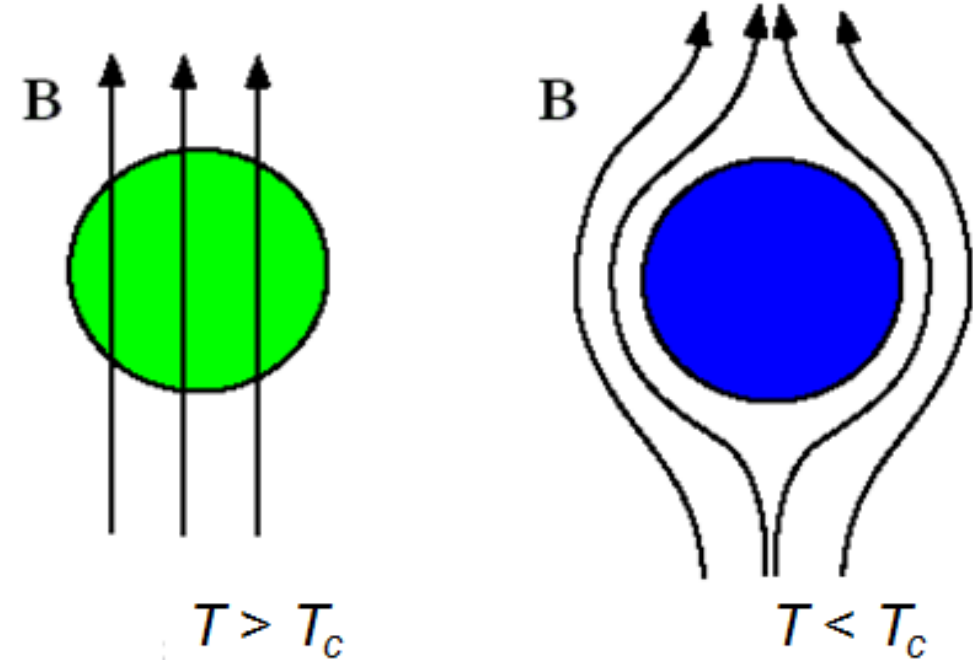
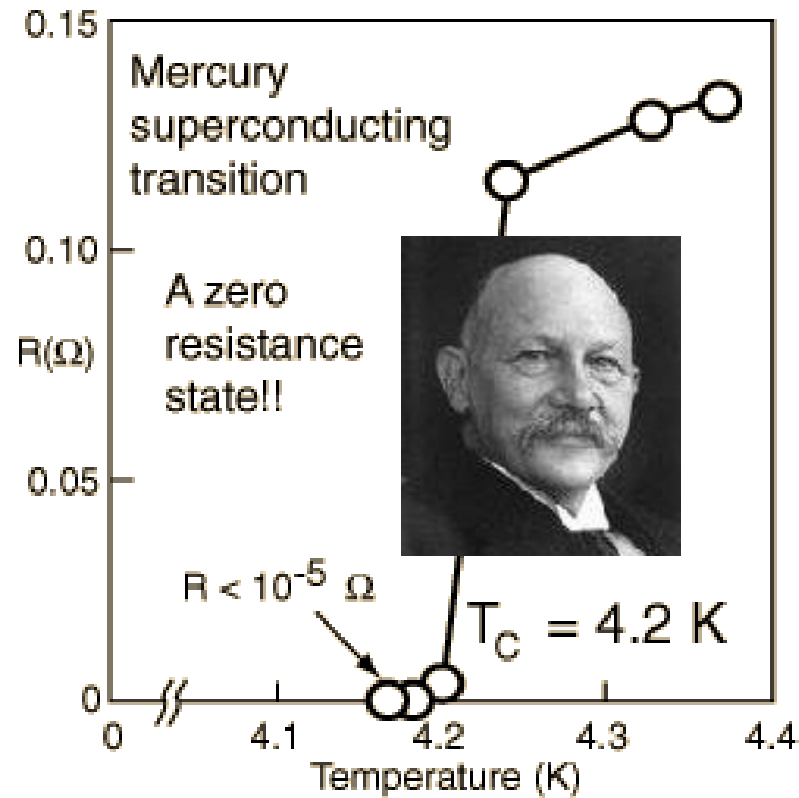


# Theory of Superconductivity: Three Milestones

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- A two-fluid model (Fritz and Heinz London, 1935)
- A phenomenological 'quantum' theory based on Landau's previously-established theory of second-order phase transitions (Vitaly Ginzburg and Lev Landau, 1950)
- A quantum BCS theory which describes superconductivity as a microscopic effect caused by a condensation of Cooper pairs into a boson-like state (John Bardeen, Leon Cooper, and John Schrieffer, 1957)

# Theory of Superconductivity: Two-Fluid Theory



London F., London, H. (1935). "The Electromagnetic Equations of the Supraconductor".  
*Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*. **149** (866) 71.

# Theory of Superconductivity: Two-Fluid Theory

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$$n = n_n + n_s$$

$$\mathbf{j} = \mathbf{j}_n + \mathbf{j}_s = -e(n_n \mathbf{v}_n + n_s \mathbf{v}_s)$$

The normal fluid is dissipative  $\mathbf{j}_n = \sigma_n \mathbf{E}$

The superfluid obeys the classical Newton's law of motion

$$m \frac{d\mathbf{v}_s}{dt} = -e\mathbf{E} \Rightarrow \frac{d\mathbf{j}_s}{dt} = \frac{n_s e^2}{m} \mathbf{E}$$

# Theory of Superconductivity: Two-Fluid Theory

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$$\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \qquad \nabla U = \frac{\partial U}{\partial x} \vec{i} + \frac{\partial U}{\partial y} \vec{j} + \frac{\partial U}{\partial z} \vec{k} = \text{grad} U$$

$$\vec{a}(x, y, z) = P(x, y, z) \vec{i} + Q(x, y, z) \vec{j} + R(x, y, z) \vec{k}$$

$$\text{div} \vec{a} = (\nabla, \vec{a}) = \left( \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}, P \vec{i} + Q \vec{j} + R \vec{k} \right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

$$\text{rot } \vec{a} = [\nabla, \vec{a}] = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

$$[a, [b, c]] = b(a, c) - c(a, b)$$



# Theory of Superconductivity: Two-Fluid Theory

$$\vec{j}_s = n_s e \vec{v}_s$$

**The Newton's law**

$$F = m \frac{d}{dt}(\vec{v}_s) \quad \Rightarrow \quad eE = m \frac{d}{dt} \left( \frac{\vec{j}_s}{n_s e} \right) \quad \Rightarrow \quad \boxed{E = \frac{d}{dt} (\Lambda \vec{j}_s) \quad \Lambda = \frac{m}{n_s e^2}}$$

**Faraday's law of induction**

$$\text{rot} \vec{E} = -\frac{1}{c} \frac{d\vec{B}}{dt}$$

$$\frac{n_s e^2 E}{m} = \frac{d\vec{j}_s}{dt} \quad \Rightarrow \quad \vec{\nabla} \times \frac{n_s e^2 \vec{E}}{m} = \vec{\nabla} \times \frac{d\vec{j}_s}{dt} \quad \Rightarrow \quad -\frac{n_s e^2}{mc} \frac{d\vec{B}}{dt} = \vec{\nabla} \times \frac{d\vec{j}_s}{dt}$$

$$\Rightarrow \quad \frac{d}{dt} \left[ \vec{\nabla} \times \vec{j}_s + \frac{n_s e^2}{mc} \vec{B} \right] = 0 \quad \Rightarrow \quad \boxed{\vec{\nabla} \times \vec{j}_s = -\frac{n_s e^2}{mc} \vec{B}}$$

# Theory of Superconductivity: Two-Fluid Theory

$$E = \frac{d}{dt}(\Lambda j_s) \quad \Lambda = \frac{m}{n_s e^2}$$

Ampère's circuital law

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

$$\vec{\nabla} \times \vec{j}_s = -\frac{n_s e^2}{mc} \vec{B}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{\nabla} \times \vec{j}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \vec{\nabla}^2 \vec{B} = -\frac{4\pi n_s e^2}{mc^2} \vec{B}$$

Gauss's law for magnetism  
equals zero

$$\vec{\nabla}^2 \vec{B} = \frac{4\pi n_s e^2}{mc^2} \vec{B}$$

$$\frac{d^2 B(z)}{dz^2} = \frac{4\pi n_s e^2}{mc^2} B(z)$$

The magnetic field can only penetrate up to a distance on the order of  $\lambda_L$  inside the superconductor.

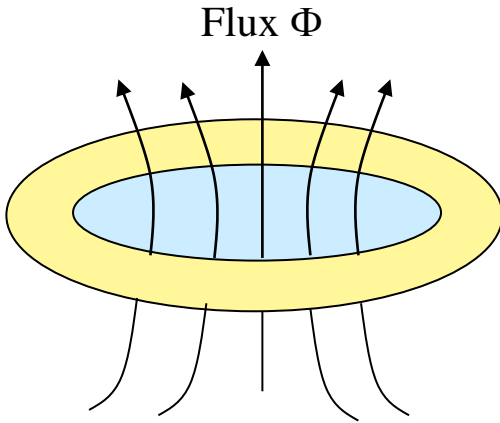
$$\lambda_L = \sqrt{\frac{mc^2}{4\pi n_s e^2}}$$

London penetration depth.

# Theory of Superconductivity: Ginzburg-Landau Theory

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## Macroscopic Quantum Effects



**Flux quantization**  $\Phi = n\Phi_0$

The basic idea behind Ginzburg-Landau theory was to write the free energy as a simple functional of the order parameter(s) of a thermodynamic system and their derivatives.

The superconducting order parameter  $\Psi(\mathbf{r})$  is a complex scalar.

# Theory of Superconductivity: Ginzburg-Landau Theory

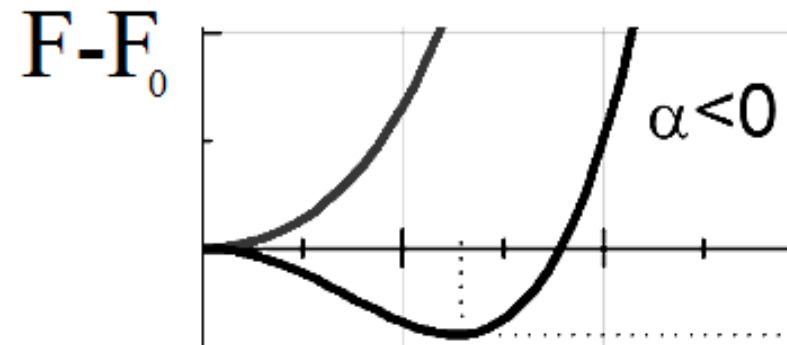
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$$F = F_0 + \alpha\eta + \frac{\beta}{2}\eta^2 \quad |T - T_c| \ll T_c$$

$$\alpha(T_c) = 0$$

$$\alpha(T) \sim T - T_c$$

$$\beta > 0$$



# Theory of Superconductivity: Ginzburg-Landau Theory

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- Without magnetic field
- The order parameter  $\psi$   
$$|\Psi(\mathbf{r})|^2 = \frac{n_s}{2}.$$
- $n_s$  is the density of Cooper pairs
- Expansion of  $f$  in powers of  $|\psi|^2$
- $|T - T_c| \ll T_c$
- $\beta > 0$ ,  $\alpha = \alpha(T)$

*Free energy of a superconductor*

$$f_s = f_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$$

Normal-state free energy

# Theory of Superconductivity: Ginzburg-Landau Theory

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With magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$f_{\text{magnetic}} = + \frac{|\mathbf{B}|^2}{8\pi}$$

$$p \rightarrow (-i\hbar\nabla - q\mathbf{A})$$

$$f_s - f_n = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \quad \rightarrow$$

$$f_s - f_n = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}|(-i\hbar\nabla - 2e\mathbf{A})\psi|^2 + \frac{|\mathbf{B}|^2}{8\pi}$$

# Task 1: Ginzburg-Landau Theory

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$$\boxed{\psi(\mathbf{r}) = |\psi(\mathbf{r})| \exp(i\varphi(\mathbf{r}))} \quad (1)$$

The free-energy density of a superconductor can be expressed in terms of the expansion in this quantity:

$$\boxed{f_s - f_n = \alpha |\psi(\mathbf{r})|^2 + \beta |\psi(\mathbf{r})|^4 + \frac{1}{2m^*} \left| \left( \frac{\hbar}{i} \nabla + \frac{e^*}{c} \mathbf{A} \right) \psi(\mathbf{r}) \right|^2} \quad (2)$$

where the subscripts n and s refer to the normal and superconducting states, respectively,  $\mathbf{A}$  is the magnetic vector potential, the magnetic field

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \quad (3)$$

# Task 1: Ginzburg-Landau Theory

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The free energy (2) is a functional of the order parameter  $\psi(\mathbf{r})$ , meaning the actual value of the order parameter realized in equilibrium satisfies  $\delta f / \delta \psi(\mathbf{r}) = 0$ .

- (a) Show that for a uniform system in a zero field a minimum of the free energy with a nonzero value becomes possible when  $\alpha$  changes sign.

In the GL theory we are interested in the region near the critical temperature of the superconducting-normal transition  $T_c$ . Thus we may take only the leading terms in the Taylor series expansions in this region:  $\alpha(T) = \alpha_0(T - T_c)$  and  $\beta = \text{constant}$ .

- (b) Find the differences in the free energies  $f_s$  and  $f_n$  below and above  $T_c$  and show that the transition to the superconducting state is energetically favorable below  $T_c$ .



# Task 1: Ginzburg-Landau Theory

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- (c) Show that the third term in Eq. (2) represents a kinetic energy of the system and equals to

$$f_{\text{kin}} = \frac{1}{2} m^* n_s^* v_s^2,$$

where  $n_s^* = \frac{1}{2} n_s$  and the superfluid velocity  $\mathbf{v}_s = \frac{1}{m^*_s} (\nabla \varphi(\mathbf{r}) + \frac{e^*}{c} \mathbf{A})$

- (d) Compare the obtained result with the general formula for the probability current in

quantum mechanics  $\mathbf{j} = \frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*)$

# Task 1: Ginzburg-Landau Theory

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Next, we obtain  $\nabla \times \mathbf{j}_s = -\frac{e^{*2} n_s^*}{m^* c} \mathbf{B}$  and recalling the Maxwell equation  $\mathbf{j}_s = \frac{c}{4\pi} \nabla \times \mathbf{B}$  we get

$$\lambda_L \nabla^2 \mathbf{B} = \mathbf{B} \quad (4)$$

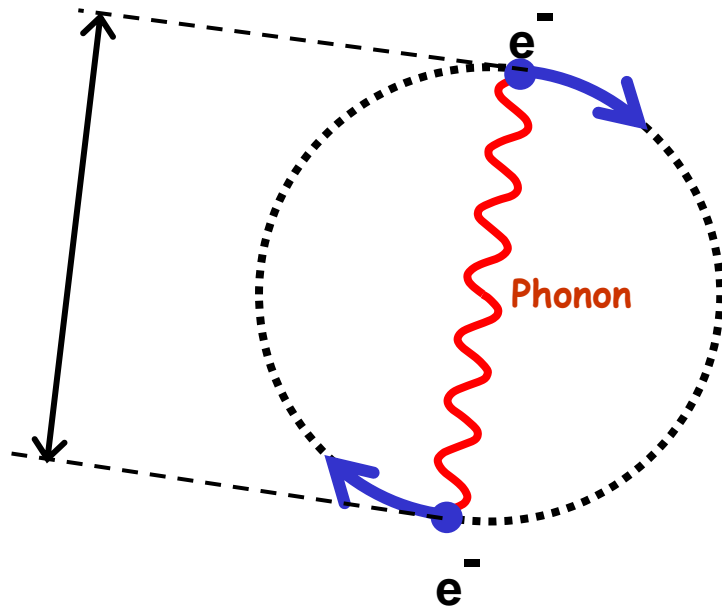
where  $\lambda_L$  is known as a London penetration depth.

- (e) Find the expression connecting  $\lambda_L$  with  $m^*$ ,  $e^*$  and  $n_s^*$ , analyze the temperature behavior of the magnetic penetration depth using the temperature dependence of the parameter  $\alpha$ .

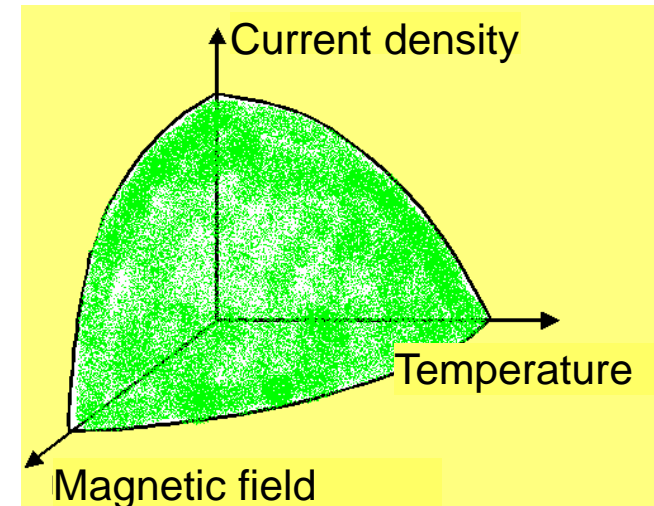
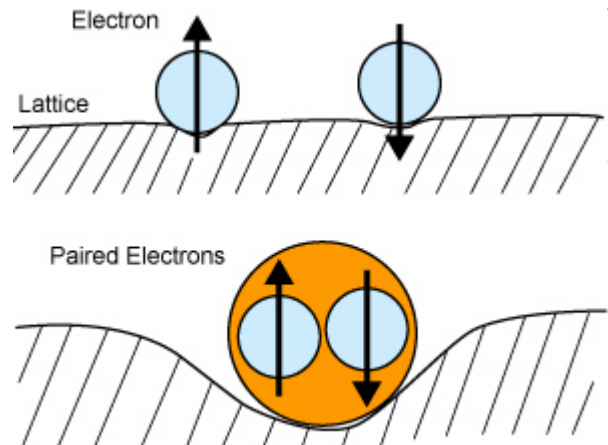
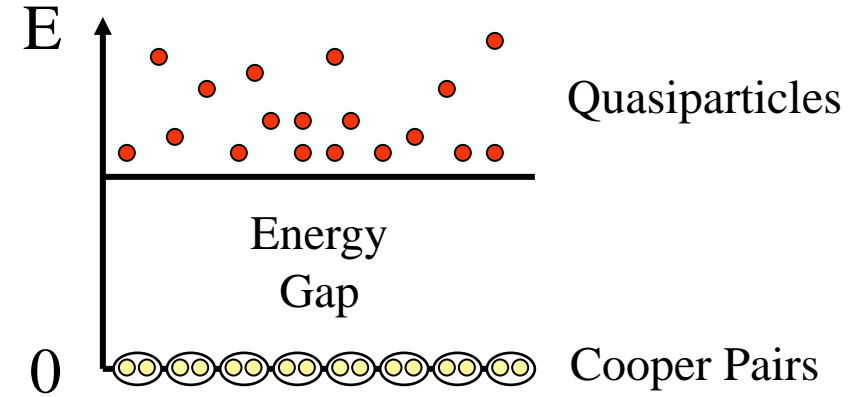
If one considers a superconducting half-space, i.e., a superconductor for  $x > 0$ , and weak external magnetic field  $\mathbf{B}$  applied along  $z$  direction in the empty space  $x < 0$ , then inside the superconductor the magnetic field decays exponentially with the characteristic length scale given by  $\lambda_L$ .

- (f) Prove the latter statement which is known as a Meissner effect. Analyze the temperature dependence of the magnetic penetration depth using the temperature dependence of the parameter  $\alpha$ .

# Theory of Superconductivity: BCS Theory



Cooper pair



# Theory of Superconductivity: BCS Theory

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Consider a pair of electrons in a singlet state which is described by the following wave function

$$\psi(\mathbf{r}_1 - \mathbf{r}_2) = \int \frac{d\mathbf{k}}{(2\pi)^3} \chi(\mathbf{k}) \exp(i\mathbf{k}(\mathbf{r}_1 - \mathbf{r}_2)) \quad (1)$$

The Schrödinger equation for two electrons interacting via the potential  $V$  reads as

$$\left[ -\frac{\hbar^2}{2m} (\nabla_1^2 + \nabla_2^2) + V(\mathbf{r}_1 - \mathbf{r}_2) \right] \psi(\mathbf{r}_1 - \mathbf{r}_2) = E \psi(\mathbf{r}_1 - \mathbf{r}_2). \quad (2)$$

Here the energy eigenvalue  $E$  is defined relative the Fermi level  $2E_F$ .

(a) Show that in the momentum representation the Schrödinger equation takes the form

$$\left( E - 2\frac{\hbar^2 k^2}{2m} \right) \chi(\mathbf{k}) = \int \frac{d\mathbf{k}'}{(2\pi)^3} V(\mathbf{k}, \mathbf{k}') \chi(\mathbf{k}') \quad (3)$$

# Theory of Superconductivity: BCS Theory

We suppose a simplest attractive form for the attractive potential of the pair

$$V(\mathbf{k}, \mathbf{k}') = \begin{cases} -V, & E_F \leq \frac{\hbar^2 k_1^2}{2m}, \frac{\hbar^2 k_2^2}{2m} \leq E_F + \hbar\omega \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

where  $\omega$  is the upper frequency of the phonon spectrum and look for a bound state with the energy  $E$  less than  $2E_F$ . The binding energy will be

$$\Delta = 2E_F - E. \quad (6)$$

(b) Show that a bound state of energy  $E$  exists provided by the expression

$$V \int_{E_F}^{E_F + \hbar\omega} \frac{N(E') dE'}{2E' - E} = 1 \quad (7)$$

where  $N(E)$  is the density of one-electron levels of a given spin.

c) Show that Eq. (7) has the solution with  $E < 2E_F$  for arbitrarily weak  $V$  in the case if  $N(E_F)$  is not zero.

d) Assuming that  $N(E) = N(E_F)$  for  $E_F \leq E \leq E_F + \hbar\omega$  show that the binding energy is given

by 
$$\Delta = 2\hbar\omega \frac{\exp(-2/N(E_F)V)}{1 - \exp(-2/N(E_F)V)} \approx 2\hbar\omega \exp(-2/N(E_F)V) \text{ for a weak attractive potential.}$$